THE APRIL MEETING AT STANFORD UNIVERSITY

The three hundred seventy-seventh meeting of the American Mathematical Society was held at Stanford University, California, on Saturday, April 5, 1941. The attendance was about sixty including the following thirty-nine members of the Society:

H. M. Bacon, Clifford Bell, B. A. Bernstein, H. F. Blichfeldt, J. H. Bowman, D. G. Chapman, G. B. Dantzig, T. C. Doyle, G. C. Evans, A. L. Foster, E. C. Goldsworthy, C. A. Hayes, Lulu Hofmann, D. H. Lehmer, R. J. Levit, J. V. Lewis, C. F. Luther, Rhoda Manning, W. A. Manning, A. E. Marston, C. B. Morrey, A. P. Morse, W. H. Myers, M. L. Nelson, Jerzy Neyman, A. R. Noble, W. T. Puckett, T. M. Putnam, R. M. Robinson, E. B. Roessler, A. C. Schaeffer, W. H. Simons, Pauline Sperry, L. H. Swinford, Gabor Szegö, A. W. Tucker, J. V. Uspensky, Morgan Ward, W. F. Whitmore.

General sessions were held in the morning and in the afternoon. The meeting was called to order at 10 A.M., and was presided over at different times by Professors Gabor Szegö, G. C. Evans, and T. M. Putnam. By invitation of the Program Committee, Professor H. J. Stewart of the California Institute of Technology delivered an hour address on *Hydrodynamic problems arising from the investigation of the transverse circulation in the atmosphere.*

Following the meeting Professor and Mrs. Szegö served tea at their home for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below. Abstracts whose numbers are followed by *t* were read by title. Dr. William Mersman was introduced by Professor E. B. Roessler.

1. L. H. Swinford: *On Abel's equation.* (Abstract 47-5-250.)
2. William Mersman: *Heat conduction in an infinite composite solid.* (Abstract 47-5-256.)
3. D. H. Lehmer: *On the values of certain Hurwitzian continued fractions.* (Abstract 47-5-213.)
4. A. W. Tucker: *Simplicial bands.* (Abstract 47-5-286.)
5. Morgan Ward: *Finite operators of formal power series.* (Abstract 47-5-206.)
6. G. B. Dantzig: *Variance of the error of a mean computed by grouping.* (Abstract 47-5-269.)
7. W. T. Puckett: *Concerning a transformation in the plane.* Preliminary report. (Abstract 47-5-282.)
8. Nathaniel Coburn: *Frenet formulas for curves in unitary space.* (Abstract 47-5-263-*t*.)
9. Henry Scheffé: *Note on the reduction of χ^2 for fit of a frequency distribution.* (Abstract 47-5-274-*t*.)

10. Hans Lewy: *Fractional potentials.* (Abstract 47-5-231-*t*.)

11. H. M. Schwartz: *Sequences of Darboux-Stieltjes integrals.* (Abstract 47-5-246-*t*.)

12. R. K. Wakerling: *On the rational loci of* $\infty^1(\rho-1)$*-spaces in r-space.* (Abstract 47-5-267-*t*.)

13. Max Zorn: *Transcendental p-adic numbers related to roots of unity.* (Abstract 47-5-215-*t*.)

T. M. PUTNAM,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The three hundred seventy-eighth meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 11–12, 1941. The attendance was about two hundred thirty including the following one hundred eighty-seven members of the Society:

A. A. Albert, G. E. Albert, W. L. Ayres, Reinhold Baer, R. M. Ballard, R. H. Bardell, R. W. Barnard, I. A. Barnett, Walter Bartky, E. F. Beckenbach, J. H. Bell, D. L. Bernstein, S. F. Bibb, D. H. Blackwell, G. A. Bliss, Morris Bloom, Henry Blumberg, L. M. Blumenthal, H. R. Brahana, Richard Brauer, Leonard Bristow, F. L. Brooks, H. K. Brown, O. E. Brown, P. B. Burcham, R. S. Burington, Herbert Busemann, L. E. Bush, J. W. Calkin, E. J. Camp, H. H. Campaigne, A. D. Campbell, E. W. Chittenden, R. V. Churchill, C. E. Clark, M. D. Clement, L. W. Cohen, B. H. Colvin, A. H. Copeland, J. J. Corliss, L. L. Cronvich, E. L. Crow, D. R. Curtiss, J. H. Curtiss, W. M. Davis, M. M. Day, J. J. DeCicco, L. L. Dines, J. M. Dobbie, D. M. Dribin, R. J. Duffin, W. D. Duthie, J. J. Eachus, J. M. Earl, Samuel Eilenberg, Benjamin Epstein, H. P. Evans, H. S. Everett, L. R. Ford, R. H. Fox, Bernard Friedman, Morris Friedman, H. L. Garabedian, E. R. Garbe, B. H. Gere, H. A. Giddings, J. S. Gold, H. H. Goldstine, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, G. J. Haltiner, R. W. Hamming, G. E. Hay, O. C. Hazlett, A. E. Heins, E. D. Hellinger, M. R. Hestenes, J. F. Heyda, J. J. L. Hinrichsen, D. L. Holl, A. S. Householder, D. H. Hyers, M. H. Ingraham, Dunham Jackson, S. B. Jackson, R. N. Johanson, Wilfred Kaplan, William Karush, I. F. Keeler, J. L. Kelley, Lois Kiefer, H. S. Kieval, S. C. Kleene, W. C. Krathwohl, A. C. Ladner, E. P. Lane, R. E. Langer, W. G. Leavitt, R. A. Leibler, C. B. Lindquist, A. T. Lonseth, A. W. McGaughey, Saunders MacLane, Ralph Mansfield, Morris Marden, R. H. Marquis, C. W. Mendel, Herman Meyer, H. L. Meyer, E. W. Miller, R. A. Miller, W. E. Milne, W. L. Mitchell, Virginia Modesitt, C. W. Moran, Vladimir Morkovin, Max Morris, E. J. Moulton, J. P. Nash, A. L. Nelson, K. L. Nielsen, I. M. Niven, E. A. Nordhaus, E. N. Oberg, J. W. Odle, Rufus Oldenburger, E. W. Paxson, J. F. Paydon, E. R. Peck, G. H. Peebles, Sam Perlis, J. E. Powell, Tibor Radó, E. D. Rainville, W. C. Randels, R. B. Rasmusen, Maxwell Reade, W. T. Reid, C. E. Rickart, R. F. Rinehart, P. G. Robinson, Arthur Rosenthal, M. A. Sadowsky, R. G. Sanger, Max Sasuly, O. F. G. Schilling,

E. R. Schneckenburger, K. C. Schraut, G. E. Schweigert, W. T. Scott, M. E. Shanks, C. G. Shover, H. A. Simmons, A. H. Smith, F. C. Smith, W. S. Snyder, E. S. Sokolnikoff, I. S. Sokolnikoff, N. E. Steenrod, R. C. Stephens, B. M. Stewart, C. F. Strobel, Otto Szász, H. P. Thielman, J. M. Thompson, C. W. Topp, L. W. Tordella, H. C. Trimble, P. L. Trump, S. M. Ulam, Henry Van Engen, G. B. Van Schaack, J. I. Vass, H. S. Wall, M. S. Webster, K. W. Wegner, M. E. Wescott, G. W. Whaples, G. W. Whitehead, L. R. Wilcox, L. A. Wolf, M. C. Wolf, F. E. Wood, M. A. Woodbury, J. W. T. Youngs.

The meeting opened on Friday morning with three sections, Analysis, Professor E. W. Chittenden presiding; Analysis and Applied Mathematics, Professor R. V. Churchill presiding; and Algebra, Professor A. A. Albert presiding. On Friday afternoon Professor M. R. Hestenes gave the Symposium Lecture under the title *The problem of Bolza in the calculus of variations*. At this lecture Professor G. A. Bliss presided. On Saturday morning there were two sections, Analysis, Professor E. F. Beckenbach presiding; and Geometry and Topology, Professor L. M. Blumenthal presiding.

All sessions met in Eckhart Hall and the adjoining Ryerson Laboratory.

Friday afternoon after the Symposium Lecture the ladies of the Department of Mathematics of the University of Chicago served tea in the Commons Room of Eckhart Hall.

The dinner on Friday evening at Burton Court was attended by one hundred sixty-eight mathematicians and guests. The toastmaster, Professor Tibor Radó, pointed out that the University of Chicago was soon to have another claim to fame in having the youngest looking Professor Emeritus in the world in the person of Professor G. A. Bliss, who retires from the faculty at the end of the coming summer quarter. He then called on two former students of Professor Bliss, Professors L. L. Dines and L. M. Graves, who related many interesting incidents in the teaching career of Professor Bliss. Pro-Professor Bliss then spoke of the earlier mathematicians at the University of Chicago and of his pleasure in being associated with them. Many former students of Professor Bliss, who could not be present, sent telegrams and letters of greetings which were presented to him at the dinner.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers numbered 1 to 8 were read before the Analysis section Friday morning, papers 9 to 15 before the section for Analysis and Applied Mathematics, papers 16 to 26 before the Algebra section, papers 27 to 33 before the Analysis section on Saturday morning, and papers 34 to 41 before the section for Geometry

and Topology. Papers 42 to 58, whose abstract numbers are followed by the letter *t*, were read by title. Mr. W. R. Scott was introduced by Professor Tibor Radó, Dr. R. S. Pate by Professor H. R. Brahana, Mr. L. J. Savage by Professor S. B. Myers, and Miss Harlan C. Miller by Professor R. L. Moore. Paper 31 was read by Professor Wall, 32 by Dr. Scott, and 39 by Dr. Youngs.

1. D. H. Hyers: *On the stability of the linear functional equation.* (Abstract 47-3-134.)

2. C. E. Rickart: *Integration in a convex linear topological space.* (Abstract 47-5-241.)

3. W. C. Randels: *On Bessel's inequality in abstract spaces.* (Abstract 47-5-240.)

4. H. H. Goldstine: *The parametric problem of the calculus of variations in general analysis.* (Abstract 47-5-224.)

5. W. R. Scott: *A lemma on the Weierstrass E-function.* Preliminary report. (Abstract 47-5-247.)

6. E. F. Beckenbach: *On functions having subharmonic logarithms.* (Abstract 47-5-216.)

7. Tibor Radó: *On the semi-continuity of double integrals in the calculus of variations.* (Abstract 47-3-140.)

8. J. F. Heyda: *A uniqueness property of general monogenic functions.* (Abstract 47-5-226.)

9. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* II. *The simple circuit.* (Abstract 47-5-255.)

10. G. E. Hay: *The finite displacement of thin rods.* (Abstract 47-5-253.)

11. I. S. Sokolnikoff: *Deflection of non-isotropic elastic plates.* Preliminary report. (Abstract 47-5-257.)

12. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* Preliminary report. (Abstract 47-5-254.)

13. E. L. Crow: *The expansion problem associated with an ordinary differential equation of first order in which the coefficient is a quadratic polynominal in the parameter.* (Abstract 47-5-221.)

14. A. T. Lonseth: *Dirichlet's principle for certain nonlinear elliptic equations.* (Abstract 47-5-232.)

15. G. H. Peebles: *On the behavior of series of polynomials orthogonal with respect to a weight function of changing sign.* (Abstract 47-5-238.)

16. H. P. Thielman: *Groups of linear fractional transformations.* (Abstract 47-5-203.)

17. M. A. Woodbury: *On ordered groups.* (Abstract 47-5-209.)

18. L. R. Wilcox: *Extensions of semi-modular lattices.* (Abstract 47-5-208.)

19. I. M. Niven: *Sums of nth powers of quadratic integers.* (Abstract 47-5-193.)

20. Reinhold Baer: *A unified theory of projective spaces and finite abelian groups.* (Abstract 47-5-182.)

21. B. M. Stewart: *Left-associated matrices with elements in an algebraic domain.* (Abstract 47-5-202.)

22. L. W. Griffiths: *The minimum number of variables in universal functions of polygonal numbers.* (Abstract 47-5-212.)

23. R. S. Pate: *Rings with multiple-valued operations.* (Abstract 47-5-194.)

24. H. A. Simmons: *Classes of maximum numbers associated with symmetric equations in n reciprocals.* IV. (Abstract 47-5-199.)

25. J. H. Bell: *Topics related to the factorization of matrices.* (Abstract 47-3-115.)

26. Richard Brauer: *On the connection between the ordinary and the modular characters of groups of finite order.* (Abstract 47-5-183.)

27. Benjamin Epstein: *Asymptotic problems associated with a Laplace-Mellin integral equation.* Preliminary report. (Abstract 47-5-223.)

28. R. J. Duffin: *Möbius transforms and Fourier transforms.* (Abstract 47-5-222.)

29. Dunham Jackson: *Generalization of a theorem of Korous.* (Abstract 47-5-227.)

30. Otto Szász: *On convergence and summability of trigonometric series.* (Abstract 47-5-251.)

31. W. T. Scott and H. S. Wall: *The transformation of series and sequences.* (Abstract 47-3-144.)

32. W. T. Scott and H. S. Wall: *Gronwall summability.* (Abstract 47-5-248.)

33. P. B. Burcham: *Some inclusion relations in the Hausdorff difference matrix.* (Abstract 47-5-219.)

34. V. G. Grove: *The transformation T of congruences.* (Abstract 47-5-264.)

35. J. J. DeCicco: *The differential geometry of the Laguerre group* G_6. (Abstract 47-3-154.)

36. L. J. Savage: *Distance spaces.* (Abstract 47-5-266.)

37. M. E. Shanks: *The space of all metrics on a compact space.* (Abstract 47-5-284.)

38. Samuel Eilenberg: *Banach space methods in topology.* I. (Abstract 47-5-278.)

39. G. E. Albert and J. W. T. Youngs: *The structure of locally connected topological spaces.* (Abstract 47-5-277.)

40. G. E. Schweigert: *Border transformations.* II. (Abstract 47-5-283.)

41. G. W. Whitehead: *Homotopy properties of rotation groups.* (Abstract 47-5-289.)

42. Edward Kasner and J. J. DeCicco: *Infinite groups generated by equilong involutions and symmetries.* (Abstract 47-3-160-*t*.)

43. J. J. DeCicco: *Lineal element transformations which preserve dual-isothermal families.* (Abstract 47-3-153-*t*.)

44. P. C. Hammer: *On fitting linear functions when all variables are subject to error.* (Abstract 47-5-272-*t*.)

45. Fred Kiokemeister: *The parastrophic criterion for the factorization of primes.* (Abstract 47-5-187-*t*.)

46. R. E. Lane: *The first three rational operations.* (Abstract 47-5-229-*t*.)

47. C. D. Olds: *Expansion of an arbitrary function in terms of certain polynomials.* (Abstract 47-5-237-*t*.)

48. C. D. Olds: *On some arithmetical identities.* (Abstract 47-5-214-*t*.)

49. G. E. Albert: *On separation spaces.* Preliminary report. (Abstract 47-5-276-*t*.)

50. H. A. Simmons: *Use of matrices in solving linear Diophantine equations.* (Abstract 47-5-200-*t*.)

51. Henry Scheffé: *Linear differential equations with two-term recurrence formulas.* (Abstract 47-5-244-*t*.)

52. Reinhold Baer: *Homogeneity of projective planes.* (Abstract 47-5-261-*t*.)

53. Lincoln La Paz: *Double integral variation problems with prescribed transversality coefficients.* (Abstract 47-5-230-*t*.)

54. Harlan C. Miller: *Concerning compact unicoherent continua.* (Abstract 47-5-281-*t*.)

55. N. E. Steenrod and A. W. Tucker: *Real n-quadrics as sphere-bundles.* (Abstract 47-5-285-*t*.)

56. D. M. Dribin: *A theorem on sets of ideals in solvable extensions of algebraic number fields.* (Abstract 47-5-186-*t*.)

57. Sam Perlis: *A characterization of the radical of an algebra.* (Abstract 47-5-195-*t*.)

58. W. T. Reid: *Green's lemma and related results.* (Abstract 47-7-313-*t*.)

W. L. AYRES,
Associate Secretary

THE MAY MEETING IN WASHINGTON

The three hundred seventy-ninth meeting of the American Mathematical Society was held at The George Washington University, Washington, D. C., on Friday and Saturday, May 2–3, 1941. The attendance was approximately two hundred fifty including the following one hundred eighty-two members of the Society:

C. R. Adams, R. P. Agnew, Leonidas Alaoglu, C. B. Allendoerfer, P. H. Anderson, H. A. Arnold, J. V. Atanasoff, Y. K. Bal, Alfred Basch, P. T. Bateman, P. G. Bergmann, E. E. Betz, G. D. Birkhoff, Archie Blake, R. P. Boas, H. F. Bohnenblust, T. A. Botts, D. G. Bourgin, S. G. Bourne, R. H. Bruck, R. S. Burington, E. W. Cannon, Abraham Charnes, A. B. Coble, I. S. Cohen, R. M. Cohn, J. B. Coleman, Richard Courant, W. H. H. Cowles, M. J. Cox, M. T. Curran, H. B. Curry, Tobias Dantzig, C. H. Dowker, Arnold Dresden, F. G. Dressel, R. F. Dressler, J. C. Durand, Samuel Eilenberg, Benjamin Epstein, Paul Erdös, F. A. Ficken, E. J. Finan, M. M. Flood, Tomlinson Fort, R. M. Foster, F. H. Fowler, K. O. Friedrichs, T. C. Fry, Abe Gelbart, J. H. Giese, M. A. Girshick, J. W. Givens, Michael Goldberg, H. S. Grant, J. A. Greenwood, T. N. E. Greville, D. W. Hall, N. A. Hall, P. R. Halmos, W. J. Harrington, G. A. Hedlund, M. H. Heins, Edward Helly, H. C. Hicks, Einar Hille, M. P. Hollcroft, T. R. Hollcroft, Witold Hurewicz, Nathan Jacobson, R. F. Johnson, F. E. Johnston, H. A. Jordan, Mark Kac, Shizuo Kakutani, J. H. Kenna, R. B. Kershner, B. F. Kimball, J. W. Kitchens, J. R. Kline, H. N. Laden, O. E. Lancaster, A. E. Landry, M. E. Layne, Solomon Lefschetz, Joseph Lehner, Marie Litzinger, E. R. Lorch, Eugene Lukacs, D. T. McClay, N. H. McCoy, E. J. McShane, J. K. L. MacDonald, C. C. MacDuffee, H. M. MacNeille, Dorothy Maharam, P. T. Maker, Dorothy Manning, M. H. Martin, F. M. Mears, Don Mittleman, Elizabeth Monroe, T. W. Moore, Marston Morse, F. D. Murnaghan, F. J. Murray, F. G. Myers, Yael Naim, K. L. Nielsen, Ivan Niven, F. W. Owens, H. B. Owens, J. C. Oxtoby, Gordon Pall, G. W. Patterson, Everett Pitcher, George Polya, H. A. Rademacher, Tibor Radó, C. J. Rees, R. G. D. Richardson, E. K. Ritter, H. E. Robbins, J. H. Roberts, H. P. Robertson, M. S. Robertson, L. B. Robinson, W. J. Robinson, J. B. Rosser, Raphaël Salem, S. A. Schelkunoff, Ivor Schilansky, Abraham Schwartz, W. M. Scott, I. E. Segal, Abraham Seidenberg, I. M. Sheffer, Seymour Sherman, J. A. Shohat, Abraham Sinkov, M. F. Smiley, T. L. Smith, Virgil Snyder, V. E. Spencer, J. J. Stoker, M. H. Stone, J. D. Tamarkin, A. H. Taub, J. H. Taylor, B. J. Tepping, M. E. Terry, J. M. Thompson, C. J. Thorne, R. M. Thrall, E. W. Titt, C. B. Tompkins, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, J. H. Van Vleck, Oswald Veblen, T. L. Wade, G. L. Walker, A. D. Wallace, Henry Wallman, G. C. Webber, F. M. Weida, C. H. Wheeler, P. A. White, A. L. Whiteman, G. T. Whyburn, Norbert Wiener, J. D. Williams, M. H. Williams, W. L. G. Williams, John Williamson, H. A. Wood, D. W. Woodard, J. W. Wrench, J. W. T. S. Youngs, Oscar Zariski, Antoni Zygmund.

The meeting opened Friday morning in two sections, Analysis, Professor Tibor Radó presiding, and Algebra and Geometry, Professor Gordon Pall and Dr. R. M. Thrall presiding.

Two general sessions were held Friday afternoon. At the first, Vice President F. D. Murnaghan presiding, Professor F. J. Murray of

Columbia University gave an address entitled *The analysis of linear transformations*. The second, Professor A. B. Coble presiding, was for short papers on various subjects.

On Saturday morning at a general session, Professor J. A. Shohat presiding, Professor I. M. Sheffer of The Pennsylvania State College gave an address entitled *Some applications of certain polynomial classes*. This was followed by two sections, Algebra and Topology, Professor C. C. MacDuffee presiding, and Probability, Statistics, Applied Mathematics, Professor Norbert Wiener presiding.

Saturday afternoon was devoted to a symposium on the Rayleigh-Ritz method and its applications, at which President Marston Morse presided. The following addresses were given: *Variational methods for the solution of boundary value problems* by Professor Richard Courant of New York University; *Engineering applications of the Rayleigh-Ritz method* by Professor J. P. Den Hartog of Harvard University; *Applications of the Rayleigh-Ritz method in the theory of the structure of matter* by Professor H. M. James of Purdue University. The discussion was led by Professor J. H. Van Vleck of Harvard University and Professor P. M. Morse of Massachusetts Institute of Technology.

All sessions were held in the Hall of Government of The George Washington University.

On Friday afternoon tea was served at Columbian House.

The dinner Friday evening at the Cosmos Club was attended by one hundred thirty members and guests. The toastmaster, Professor C. R. Adams, introduced three speakers, Professor R. W. Bolwell, Chairman of the Graduate Council of The George Washington University, and Professors C. C. MacDuffee and H. P. Robertson.

At the beginning of the symposium Saturday afternoon Professor Gordon Pall presented a resolution expressing the appreciation and thanks of the Society to The George Washington University, the members of the local committee, and all others concerned for the excellent arrangements for the meeting. The resolution was unanimously adopted by a rising vote.

The Council met at 9:00 P.M., Friday, May 2, in the Holmes Room of the Cosmos Club.

The Secretary announced the election of the following fifty-nine persons to membership in the Society:

Mr. Milton Abramowitz, Brooklyn, N. Y.;
Miss Grace Karen Anderson, American Mathematical Society, New York, N. Y.;
Professor Tomás R. Bachiller, University of Madrid, Madrid, Spain;
Mr. Yusuf Kenan Bal, Columbia University;

Mr. James Douglas Bankier, Rice Institute;
Mr. Paul Trevier Bateman, University of Pennsylvania;
Professor Maurice Anthony Biot, Columbia University;
Professor Juan Blaquier, University of Buenos Aires, Buenos Aires, Argentina;
Professor Clotilde Bula, University of Rosario, Rosario, Argentina;
Mr. Philip Staats Carter, R.C.A. Communications Inc., Rocky Point, L. I., N. Y.;
Mr. Leo Cohen, Columbia University;
Mr. Charles Raymond DePrima, New York University;
Dr. Henry Chabot Dieckmann, Occidental College, Los Angeles, Calif.;
Dean Margaret Criswell Disert, Wilson College, Chambersburg, Pa.;
Mr. Robert Franklin Dressler, University of Pennsylvania;
Professor William Maurice Ewing, Lehigh University;
Mr. W. Eugene Ferguson, University of Missouri;
Professor Fernando L. Gaspar, University of Rosario, Rosario, Argentina;
Dr. John H. Giese, Rutgers University;
Dr. Basil E. Gillam, University of Missouri;
Mr. Thomas Franklin Glass, Rice Institute;
Dean John Alonzo Goff, Towne Scientific School, University of Pennsylvania;
Mr. Richard Albert Good, University of Wisconsin;
Mr. Lewis Greenwald, New York Evening High School;
Mr. Eli Grossman, United States Life Insurance Co., New York, N. Y.;
Professor Emil J. Gumbel, New School for Social Research, New York, N. Y.;
Professor Harrod Harold Hartzler, Goshen College, Goshen, Ind.;
Professor Clifton T. Hazard, Purdue University;
Dr. Aaron Paul Horst, Procter and Gamble Co., Cincinnati, Ohio;
Dr. Colin Munroe Hudson, Frankford Arsenal, Philadelphia;
Dr. Leroy Charles Hutchinson, Michigan College of Mining and Technology, Houghton, Mich.;
Professor Ralph William Jones, University of Delaware;
Major Paul Hanes Kemmer, Chief Aircraft Laboratory, Wright Field, Dayton, Ohio;
Dr. John Maurice Kingston, University of Washington;
Mr. Joseph Landin, New York University;
Miss Jeanne Starrett Le Caine, Cambridge, Mass.;
Mr. Max LeLeiko, Willard Contracting Corporation, New York, N. Y.;
Dr. Eugene Lukacs, Baltimore, Md.;
Dr. Rudolf Luneburg, Spencer Lens Company, Buffalo, N. Y.;
Dr. Szolem Mandelbrojt, Rice Institute;
Dr. Warren Perry Mason, Bell Telephone Laboratories, New York, N. Y.;
Dr. William Alvin Mersman, University of California, College of Agriculture;
Mr. Osborn Maitland Miller, American Geographical Society, New York, N. Y.;
Mr. Olen Alvin Nance, University of Missouri;
Dr. William Donald Phelps, R.C.A. Manufacturing Co., Camden, N. J.;
Mr. Harry Pollard, Dorchester, Mass.;
Dr. Virgil Nelson Robinson, Louisiana State University;
Professor Frederick William Rogers, University of Cincinnati;
Mr. Jerome Colbert Smith, University of Pennsylvania;
Dr. Ernst Snapper, Princeton University;
Mr. Herbert Solomon, Columbia University;
Dr. Thomas H. Southard, Wayne University;

Mr. Wolfgang Joseph Thron, Rice Institute;
Dr. Philip Russell Wallace, University of Cincinnati;
Mr. Edward Faith White, University of Cincinnati;
Mr. Albert Wiesenberg, Columbia University;
Mr. Herbert Wolfe, Brooklyn College (Evening Session);
Mr. Richard Schuyler Wolfe, Great Falls Commercial College, Great Falls, Mont.;
Mr. Max A. Woodbury, University of Michigan.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of Chicago: Messrs. William Caswell Carter, Gerhard Karl Kalisch, and Richard Donald Schafer, and Miss Alice Elizabeth Turner.
Lehigh University: Mr. Leonard Pattillo Burton.
Metropolitan Life Insurance Company: Messrs. Malvin E. Davis and James Russell Herman.
Syracuse University: Mr. William John Strange.
Wayne University: Mr. Clifford Raymond Simms.
Wesleyan University: Messrs. Francis Escott and Joseph Wannemacher.

The following appointments by President Marston Morse were reported: as a standing Committee on Places of Meetings, Professors A. B. Coble (chairman), 3 years, W. B. Carver, 2 years, and E. J. McShane, 1 year; as auditors of the Society's books for 1941, Professors R. G. Archibald and A. E. Meder, Jr.; as an additional member of the Committee on Arrangements for the Annual Meeting of 1941, Professor W. D. Cairns; as Committee on Arrangements for the Summer Meeting of 1942 at Cornell University, Professors R. P. Agnew (chairman), W. D. Cairns, A. D. Campbell, T. R. Hollcroft, J. B. Rosser, R. J. Walker; as chairman of the Nominating Committee, to replace Professor G. A. Bliss, Professor A. B. Coble; as representative of the Society at the inauguration of Victor Raymond Edman as fourth President of Wheaton College, Wheaton, Illinois, on May 9, 1941, Professor L. M. Graves; as representative of the Society at the inauguration of Virgil Melvin Hancher as President of The State University of Iowa on May 24, 1941, Professor R. B. McClenon; as representative of the Society at the Fiftieth Anniversary Celebration of the University of Chicago on September 27, 28, and 29, 1941, Professor D. R. Curtiss. It was also reported that President Marston Morse will represent the Society at the Centenary Celebration of Fordham University on September 15–17, 1941.

The dates for the Annual Meeting of 1941 at Lehigh University were set as December 29–31, 1941.

The Secretary reported that Professor E. J. McShane had accepted the invitation to deliver the Colloquium Lectures at the Summer

Meeting of 1943; his subject will be *Existence theorems in the calculus of variations.*

The Secretary reported that the following had been chosen as the voting representatives on the Council of the editorial committees for the year 1941: Bulletin—Dean Tomlinson Fort; Transactions—Professor C. C. MacDuffee; Colloquium—Professor J. D. Tamarkin; American Journal of Mathematics—Professor T. H. Hildebrandt.

The Secretary reported that an unrestricted gift of $1,000 had been received from Dr. Robert Henderson. The following resolution was unanimously adopted and copies were ordered spread upon the records and sent to Dr. Henderson:

> The Council of the American Mathematical Society expresses its deep gratitude to Dr. Robert Henderson for his generous gift of $1,000 to the Society. It also wishes at this time to make grateful acknowledgment of his many years of faithful service on the Board of Trustees where his wise counsel has been invaluable. We recognize these acts as the natural expression of a life devoted to the study and application of mathematics.

It was announced that Professor Oscar Zariski had been elected as a member of the Transactions Editorial Committee to fill the unexpired term of the late Professor W. C. Graustein. Professor Hille reported that Professors Nelson Dunford and R. L. Wilder had been appointed Associate Editors.

Professors J. L. Walsh (chairman), E. W. Chittenden, and E. J. McShane were appointed as a committee to confer with the Williams and Wilkins Company on an agreement concerning the proposed new series of monographs for which the name *Mathematical Surveys* has been suggested. The Council had previously voted to approve the memorandum submitted by the Company in which the introduction of this series was suggested.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 11 were read before the section for Analysis; those numbered 12 to 22 before the section for Algebra and Geometry; those numbered 23 to 27 before the general session on Friday afternoon; those numbered 28 to 34 before the section for Algebra and Topology; those numbered 35 to 41 before the section for Probability, Statistics, Applied Mathematics; those numbered 42 to 67 were read by title. Paper 8 was read by Mr. Harrington, paper 14 by Dr. Smiley, paper 17 by Dr. Bruck, paper 18

by Professor Wade, paper 27 by Professor Polya, and paper 38 by Mr. Thorne. Dr. Bergman was introduced by Dr. D. C. Spencer, Mr. Gravalos by Professor G. D. Birkhoff, Professor Sutton by Professor C. B. Allendoerfer, Dr. Weinstein by Dr. H. E. Robbins, Dr. Bers by Dr. Abe Gelbart.

1. Joseph Lehner: *A partition function connected with the modulus five.* (Abstract 47-7-298.)

2. Stefan Bergman: *The method of the minimum integral and the analytic continuation of functions.* (Abstract 47-5-217.)

3. K. L. Nielsen: *Concerning boundary value problems for linear differential equations when the boundary conditions are given by Stieltjes integrals.* (Abstract 47-5-236.)

4. E. J. McShane: *Sufficient conditions for a weak relative minimum in the problem of Bolza.* (Abstract 47-7-309.)

5. F. A. Ficken: *Certain systems of subsets of quasi and partially ordered sets.* (Abstract 47-7-328.)

6. E. R. Lorch: *The spectrum of linear transformations.* (Abstract 47-5-235.)

7. W. J. Trjitzinsky: *Singular Lebesgue-Stieltjes integral equations.* (Abstract 47-7-316.)

8. W. J. Harrington and J. B. Rosser: *A study of certain functions auxiliary to Brun's method in number theory.* (Abstract 47-5-225.)

9. D. G. Bourgin: *On reflexive Banach spaces.* (Abstract 47-7-304.)

10. M. H. Heins: *A generalization of the Aumann-Carathéodory "Starrheitssatz."* (Abstract 47-7-306.)

11. Raphaël Salem: *On trigonometrical series whose coefficients do not tend to zero.* (Abstract 47-5-243.)

12. W. M. Scott: *On matrix algebras over an algebraically closed field.* (Abstract 47-5-198.)

13. R. M. Thrall: *On Young's semi-normal representation of the symmetric group.* (Abstract 47-5-204.)

14. Everett Pitcher and M. F. Smiley: *Transitivities of betweenness.* (Abstract 47-5-196.)

15. Gordon Pall: *The construction of positive ternary quadratic forms.* (Abstract 47-3-174.)

16. D. T. McClay: *Clifford numbers.* (Abstract 47-7-322.)

17. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* I. (Abstract 47-5-184.)

18. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* II. (Abstract 47-5-185.)

19. Y. K. Bal: *Study on graphs.* Preliminary report. (Abstract 47-5-262.)

20. F. G. Gravalos: *The algebraic integrals of Hill's equations.* (Abstract 47-7-318.)

21. J. B. Rosser: *A generalization of the euclidean algorithm to several dimensions.* (Abstract 47-7-299.)

22. Mark Kac: *Remark on the distribution of values of the arithmetic function $d(m)$.* (Abstract 47-7-297.)

23. Tibor Radó: *On convergence in length and convergence in area.* (Abstract 47-5-239.)

24. J. A. Shohat: *On the best polynomial approximation for functions possessing derivatives.* (Abstract 47-5-249.)

25. C. C. MacDuffee: *Products and norms of ideals.* (Abstract 47-5-188.)

26. G. T. Whyburn: *Orbit decompositions.* (Abstract 47-5-291.)

27. George Polya and Norbert Wiener: *On the oscillation of the derivatives of a periodic function.* (Abstract 47-7-312.)

28. G. L. Walker: *The elementary divisors of a direct product of matrices.* (Abstract 47-5-205.)

29. H. A. Arnold: *Homology in set-intersections, with an application to r-regular convergence.* (Abstract 47-7-325.)

30. I. M. Niven: *Equations in quaternions.* (Abstract 47-5-192.)

31. A. D. Wallace: *A fixed-point theorem for trees.* (Abstract 47-5-287.)

32. Samuel Eilenberg: *Banach space methods in topology.* II. (Abstract 47-5-279.)

33. P. A. White: *A decomposition of true cyclic elements by means of continua.* (Abstract 47-5-288.)

34. Witold Hurewicz: *On duality theorems.* (Abstract 47-7-329.)

35. Alfred Basch: *A contribution to the theory of multiple correlation.* (Abstract 47-7-324.)

36. J. A. Greenwood: *A theorem on probability assignments to events.* (Abstract 47-5-271.)

37. K. O. Friedrichs: *On the mathematical theory of the buckling of spherical shells.* (Abstract 47-7-317.)

38. C. J. Thorne and J. V. Atanasoff: *The application of a general functional approximation method to thin plate problems.* (Abstract 47-5-259.)

39. R. M. Sutton: *An instrument for drawing confocal conics.* (Abstract 47-5-258.)

40. Alexander Weinstein: *On the vibrations of a clamped plate under tension.* (Abstract 47-5-260.)

41. Richard Courant: *On a new type of variational problems, with physical demonstrations.* (Abstract 47-5-220.)

42. R. P. Agnew: *On limits of integrals.* (Abstract 47-7-301-*t*.)

43. H. A. Arnold: *On r-regular convergence of sets.* (Abstract 47-7-326-*t*.)

44. Stefan Bergman: *A method for summation of series of orthogonal functions of two variables.* (Abstract 47-7-302-*t*.)

45. Lipman Bers: *On a generalized harmonic measure.* (Abstract 47-5-218-*t*.)

46. Leonard Carlitz: *Some interpolation formulas connected with polynomials in* $GF(p^n)$. (Abstract 47-5-210-*t*.)

47. Leonard Carlitz: *The coefficients of the reciprocal of certain series.* (Abstract 47-5-211-*t*.)

48. Leonard Carlitz: *An analogue of the Bernoulli polynomials.* (Abstract 47-7-296-*t*.)

49. A. H. Clifford: *Matrix representations of completely simple semigroups.* (Abstract 47-7-293-*t*.)

50. A. H. Copeland: *If.* (Abstract 47-5-268-*t*.)

51. J. J. DeCicco: *Equilong geometry of differential equations of first order.* (Abstract 47-3-151-*t*.)

52. Samuel Eilenberg: *Irreducible transformations onto manifolds.* (Abstract 47-5-280-*t*.)

53. G. E. Forsythe: *Cesàro summability of random variables.* (Abstract 47-5-270-*t*.)

54. Einar Hille: *Notes on linear transformations.* III. *Semi-groups on Lebesgue spaces.* (Abstract 47-7-307-*t*.)

55. C. T. Hsu: *Two samples from normal bivariate populations.* (Abstract 47-5-273-*t*.)

56. L. H. Loomis: *A simple proof of the Fatou theorem.* (Abstract 47-5-234-*t*.)

57. L. H. Loomis: *A converse to the Fatou theorem.* (Abstract 47-5-233-*t*.)

58. Saunders MacLane and O. F. G. Schilling: *A formula for the direct product of cross product algebras.* (Abstract 47-5-189-*t*.)

59. Saunders MacLane and O. F. G. Schilling: *Group extensions characterizing complete fields.* (Abstract 47-5-190-*t*.)

60. Saunders MacLane and O. F. G. Schilling: *The principal divisor theorem for function fields.* (Abstract 47-5-191-*t*.)

61. H. J. Riblet: *Certain theorems for symmetric differential functions.* (Abstract 47-5-197-*t*.)

62. M. S. Robertson: *The partial sums of multivalently star-like functions.* (Abstract 47-7-314-*t*.)

63. L. B. Robinson: *Some complete systems of semitensors.* (Abstract 47-7-315-*t*.)

64. Raphaël Salem: *On some properties of symmetrical perfect sets.* (Abstract 47-5-242-*t*.)

65. Louis Weisner: *Polynomials whose roots lie in a sector.* (Abstract 47-5-207-*t*.)

66. G. T. Whyburn: *Almost periodicity.* (Abstract 47-5-290-*t*.)

67. G. T. Whyburn: *Regular almost periodicity.* (Abstract 47-5-292-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

Théorie Analytique des Associations Biologiques. I. *Principes.* 1934, 45 pp. II. *Analyse Démographique avec Application Particulière à l'Espèce Humaine.* 1939, 149 pp. By A. J. Lotka. (Actualités Scientifiques et Industrielles, nos. 187, 780.) Paris, Hermann.

The first of these monographs is a general introduction to the second. It (the first) is primarily a non-mathematical discussion of what the author regards as two types of biological evolution:

(1) The *intra-spécifique*, that which is concerned with the changes in the aggregate of distributions associated with a given biological species, brought about by mutation, genetic variation, natural selection, etc.

(2) The *inter-spécifique*, that which deals with the changes in the numbers of individuals in each of several coexisting biological species or groups, which are brought about by different birth and death rates caused by interaction of the species with each other and with other environmental factors.

The author's main interest lies in the second type of evolution, which he discusses at some length by elaborating on the implication of chains of species regarded as food chains, and other ecological points. The problem of the variation of the numbers of individuals in a system of n species is finally formulated in a system of equations $\partial x_i/\partial t = F_i(x_1, x_2, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ where x_j is the number of individuals in the jth species at time t. A formal solution is given for the case in which the F_i are approximately homogeneous linear functions of the $x_i - c_j$ in the neighborhood of $(c_1, c_2, \cdots, c_n)$ where $(c_1, c_2, \cdots, c_n)$ are the values of $x_1, x_2, x_3, \cdots, x_n$, respectively, for some "stationary state" for which the $\partial x_i/\partial t = 0$. However, this formal solution and its properties are hardly discussed at all. Special cases of solutions for the case $n=1$ and 2 are given. For the case $n=1$, the author shows that the Malthusian and the Verhulst-Pearl-Reed logistic growth functions are obtained by respectively assuming $F_1(x)$ to be linear and then quadratic in x.

In the second monograph the author concerns himself with the dynamics of human populations in which there is assumed to be no immigration or emigration. The first chapter in this monograph begins with the fundamental equation

$$N(t) = \int_0^\omega B(t-a)p(a)da \tag{1}$$

where $B(t-a)da$ is the number of individuals born within the time interval $t-a-da$, $t-a$, $p(a)$ is the proportion of individuals born at a given time who are still alive at age a, while $N(t)$ is the number of individuals present in the population at time t, and w is the maximum age attainable. w is replaced by $+\infty$ in much of the work. Lotka goes into considerable detail in dealing with this equation, deriving formulas for age distribution and average age in the population, rates of birth and mortality, survival curves, vital index, and conditions for a stationary population. Particular emphasis is placed on the Malthusian case, namely that in which $N(t)=N_0 \exp(rt)$, and $B(t)=B_0 \exp(rt)$ where r is the relative rate of increase in the population. A more general case is considered in which $B(t)$ (and hence $N(t)$) is a linear function of a function $\phi(t)$ and its derivatives. Particular attention is paid to the logistic function obtained by solving the equation $dN/dt=a_1N+a_2N^2$.

In the second chapter Lotka takes into account the effects of the variation of fertility with respect to age, by introducing a fertility function $m(a)$ as a multiplier in the integrand of (1), where $m(a)$ is the average number of daughters per mother borne by mothers in the age interval a, $a+da$. The introduction of $m(a)$ into (1) (setting $w=+\infty$) yields the integral equation

$$B(t) = \int_0^\infty B(t-a)p(a)m(a)da \tag{2}$$

which relates the number of births of daughters to the number of births of mothers. This problem and its ramifications are studied rather extensively for the Malthusian case. A formal solution of (2) for the Malthusian case of the form $B(t)=\sum_{i=1}^{\infty} Q_i \exp(r_i t)$ is obtained, where the r_i are roots of the equation obtained by setting $B(t)=B_0 \exp(rt)$ in (2). Convergence and other fundamental questions which arise in connection with this solution are touched upon only lightly.

The third chapter deals with the composition of a population from the point of view of successive generations. Thus if N is the number of mothers in a beginning (zeroth) generation of age a, the surviving female offspring which are produced during the age interval da of mothers is $Np(a)m(a)da$, which is really the distribution of births of daughters in the first generation. The distribution $B_i(t)$ of births of daughters in the ith generation is therefore obtained by integrating $B_{i-1}(t-a)p(a)m(a)da$ from 0 to t, where $B_1(t)=Np(t)m(t)$. The total distribution of daughters $B(t)$ is found by summing the distribution functions for all generations, which yields an integral equation similar

to (2). This same approach has been used by Lotka in his work on renewal theory.

The remaining four chapters deal with special problems such as determination of marriage rates, fertility rates and other indices and measures of natural increases in a population; determination of number of maternal, paternal and complete orphans; calculation of probabilities of extinction of a given line of descent and similar problems. The solution of the last problem, which has also been treated by R. A. Fisher, is extremely ingenious. A generation function $f(x)=\sum_{i=0}^{k}\pi_i x^i$ $(\sum_{i=0}^{k}\pi_i=1)$ is set up, where π_i is the probability that a male will have exactly i sons. $f(x)$ has many remarkable properties which enable the author to deal with the problem of descendance. For example $\partial f/\partial x]_{x=0}$ yields the average number of sons per male; the coefficient of x^s in $f^r(x)$ yields the probability that r males will have a total of s sons; the coefficient of x^s in $\pi_r f^r(x)$ which is a general term in $f(f(x))=f_2(x)$ say, is the probability that a man will have r sons and s grandsons. Similar interpretations can be placed on coefficients of x in the iterated function

$$f\{f(f[\cdots])\}=f_n(x).$$

The second monograph is well illustrated throughout by applications of the theory to actual census data taken from the United States, England, France, Germany, and several other countries. Numerous charts and tables are given for comparing theory with facts. The monograph is an excellent account of results which have been obtained during the past quarter of a century in the theory of population dynamics. Most of the results are due to the author himself. Lotka has shown a great deal of ingenuity in formulating the problems mathematically and in reaching practical solutions of the problems. Those interested in applied mathematics in the field of biology will find these monographs well worth reading.

S. S. WILKS

The Variate Difference Method. By Gerhard Tintner. (Cowles Commission for Research in Economics Monograph, no. 5). Bloomington, Indiana, Principia Press, 1940. 13+175 pp.

Various mathematical statistical methods have been proposed during recent years in attempts to describe, analyze and interpret economic time series. Regression analysis and its extension to harmonic analysis, moving averages, and the variate difference method are some of the techniques which have been used. The fundamental

concept adopted in these approaches is that each element w_i $(i=1, 2, \cdots, N)$ in a time series is assumed to consist of a sum m_i+x_i where m_i is the "true economic component" and x_i is the "random component." The main purpose of the statistical analysis according to these methods is to estimate and describe these two components. There are, however, other approaches which use different conceptual models. As its title indicates, the present book presents a treatment (in ten chapters) of the variate difference method, although the author takes advantage of several opportunities to discuss other methods by way of comparison.

In the variate difference method it is assumed that the m_i are "smooth" in the sense that the differences $\Delta^{(K)}(m_i)$ of the m_i of some (finite) order K vanish, while the x_i are independent random variables. Under these assumptions the main problem as discussed by Tintner is that of deciding, on the basis of the w_i and probability theory, for what value of K, $\Delta^{(K)}(m_i)=0$. The problem is therefore one of studying the probability theory of $\Delta^{(K)}(w_i)$ $(K=0, 1, 2, \cdots)$ and making suitable significance tests. For large values of N, Tintner follows rather closely the work of Oskar Anderson, which consists in comparing the difference between the variances of two consecutive series of differences, say $\Delta^{(K)}(w_i)$ and $\Delta^{(K+1)}(w_i)$, with the variance of the difference under the assumption that $\Delta^{(K)}(m_i)=0$. The kurtosis of $\Delta^{(K)}(w_i)$ (i.e., $\mu_4' - 3\sigma^2$) is also considered in this analysis.

For the case of small N, Tintner proposes a significance test for testing the hypothesis that $\Delta^{(K)}(m_i)=0$, using the two series $\Delta^{(K)}(w_i)$ and $\Delta^{(K+1)}(w_i)$. If the original x_i are assumed to be normally and independently distributed with variance σ^2, then it can be shown that $\Delta^{(K)}(w_i)$ are normally distributed with zero means and variance $C_{2K,K}\sigma^2$. Under the same assumptions, the terms $j+r(2K+3)$ $(r=0, 1, 2, 3, \cdots)$ and terms, $j+K+1+r(2K+3)$ $(r=0, 1, 2, 3, \cdots)$ will all be normally and independently distributed. Hence σ^2 can be estimated independently from the two series thus making it possible to set up an F-test. Such an F-test can be set up for $j=1, 2, 3, \cdots, 2K+3$.

The significance test based on this method of selection, while mathematically exact under the assumptions made, is very simple but not very efficient, as the author points out, in the sense that full use is not made of the data. A more efficient test would undoubtedly be very complicated.

This book is written primarily for the economic statistician who may wish to apply the variate difference method to his own economic time series. Tintner illustrates the routine application of the method in great detail, using American Wheat-Flour prices for the period

1890–1937 as his data. The author gives a large number of tables of weights and coefficients of various kinds, critical ratios of two sums of squares of differences, formulae for selections of items in various pairs of consecutive differences which would be independent, etc., to facilitate the application of the method.

A considerable amount of historical material on the variate difference method is given. One chapter is devoted to the application of Sheppard's smoothing formulae and serial correlation, and the connetion between the variate difference method and these methods is pointed out.

Several appendices are given: Appendix I is a summary of computation formulae used in the various chapters; Appendix II is devoted to the mathematics underlying the various formulae which are used; the remaining four appendices deal with special topics such as seasonal variation, normality of the random element, etc. Author and subject matter indices are provided.

The question as to whether the variate difference method is superior to other methods of time series analysis is largely a matter of opinion. Tintner has given a very good account of the method, although it appears to the reviewer that he did not concern himself enough with the problem of estimating the m_i after it had been decided for which value of K the value of $\Delta^{(K)}(m_i)=0$. He approached the problem of determining the m_i by using the Sheppard smoothing formulae, but this, of course, is essentially another method of time series analysis.

Tintner's book is an interesting contribution to the literature of time series analysis. It is well documented by references and footnotes and reflects a great deal of work on the part of the author.

S. S. WILKS

Tensor Analysis of Networks. By Gabriel Kron. New York, Wiley, 1939. 24+635 pp. $7.50.

In the introduction to this book, the author states: "It is emphasized that this book is not written by a mathematician and is not written for mathematicians. This book is written by an engineer for engineers who are interested in learning an *organized* method of attack to analyze and synthesize electrical networks."

The first two chapters provide an extremely detailed account of the tensor and matrix notation, of the multiplication of matrices and describes the impressed voltages, currents and self- and mutual impedances of a network as components of tensors e_α, i^α and $Z_{\alpha\beta}$, re-

spectively. In Chapter III the type of electromagnetic networks considered are described as mesh, junction and orthogonal networks. It is emphasized that mesh and junction networks are special cases of orthogonal networks and are due to the "special assumptions as to the nature of the impressed quantities that are known or of the response quantities that are to be found." The primitive mesh network consisting of n individual coils, each short-circuited on itself is defined and observed to satisfy the generalized Ohm's law $e_m = Z_{mn} i^n$.

The central concept of the book is that of the transformation matrix, denoted by C. Its function is to allow one to describe any particular (mesh) network in a methodical way in terms of the primitive (mesh) network. This may be thought of as being accomplished by a linear transformation of the covariant vector e. The invariance of the power, $e_m i^m$, determines in the usual fashion the transformation law of the current vector and the invariance of Ohm's law then requires the impedances to be transformed as the components of a tensor having two covariant indices.

The theory is considerably complicated by the necessity for introducing equations of constraint which require the consideration of singular transformation matrices. A discussion of this, of the complications involved in combining two more-or-less interconnected networks into a single network, and a large number of detailed applications of the theory to specific problems occupy pages 117–323. In Chapter XIII the theory is extended to allow for alternating currents by permitting the current and potential vectors to have complex numbers as elements. The power input is now taken to be $e^*_m i^m$, where the star denotes the complex conjugate. Its assumed invariance leads to transformations involving C^*, in which case spinor indices (here denoted by the addition of a bar rather than a dot) are employed. Junction networks are next discussed, their mathematical theory being obtained from that of mesh networks by interchanging contravariant and covariant indices and adding or removing a bar. After a chapter on multielectrode-tube circuits, orthogonal networks are considered. A chapter on interlinked electric and magnetic networks precedes the introduction of a metric tensor (the inductance tensor) which is claimed to represent "the additional characteristics acquired by the electrical network owing to its linkage with an underlying magnetic network."

Chapters on Compound Networks, Symmetrical Components, and Multiple Tensors are followed by chapters on the application of the theory to the analysis and synthesis of networks. An extensive bibliography and an index conclude the book.

It is evident that a book of this kind provides difficult reading for both the mathematician and the electrical engineer. The mathematician will find the mathematical details exceedingly tedious, the derivations sometimes very obscure, and the technical terminology of the engineer unfamiliar. The engineer, on the other hand, may resent being called upon to learn a large amount of mathematical notation without the prospect of any corresponding gain in the computations he must perform. In spite of this, the book demands and is receiving attention from workers in both fields. Thus the investigation of singular transformations in tensor algebra and of tensor concepts in combinatorial topology were stimulated by this book. The introduction of concepts tending to unify special methods of approach to engineering problems will, in the long run, have an important influence on the development of engineering theory.

WALLACE GIVENS

Fourier Series and Boundary Value Problems. By R. V. Churchill. New York, McGraw-Hill, 1941. 206 pp. $2.50.

This book is a useful addition to the meager number of existing books of this general nature in English. Its major use will be as a textbook for students in engineering and the sciences interested in these topics.

The book contains no more subject matter than is implied by the title; that is, it leads up to and considers the solution of the usual several linear partial differential equations by series of trigonometric, Bessel, and Legendre functions. A considerable part of the book is devoted to an exposition of the concept of orthogonal sets of functions in general and Fourier series in particular.

The book contains some material of mathematical interest, but not very suitable to certain types of engineering students. However, it is so arranged that such material can be omitted.

From the point of view of mathematical preciseness the treatment is excellent. The book is also well planned for teaching purposes.

N. LEVINSON

The Weight Field of Force of the Earth. By William H. Roever. (Washington University Studies, New Series, Science and Technology, no. 1.) St. Louis, 1940. 84 pp. $1.50.

This monograph is an extension of the author's retiring address as Chairman of Section A of the American Association for the Advancement of Science. It deals with some statical and dynamical

phenomena of the weight field of force of the earth, and is, in general, a summary of the author's earlier investigations in this field.

The earth's weight field of force is defined by the acceleration relative to the solid part of the earth which a body resting near the surface of the earth sustains, due to gravitation of the earth and the heavenly bodies, and due to the earth's motion in the absolute system.

The first three introductory sections of the monograph deal with the laws of relative motion and the effects of a moving atmosphere. In the fourth section it is shown that the tidal effects and the results of precession and nutation are negligible as compared with the gravitational attraction of the earth and the centrifugal force due to the earth's rotation. This main part of the weight field is at rest with respect to the solid part of the earth, and it possesses a potential. In the next two sections the moments actuating the Eötvös torsion balances are derived in geometrical and analytical treatment. The seventh section and the appendix are devoted to some dynamical phenomena of the weight field which lead to differential equations that can be easily integrated.

In view of the recent revival, due to the national defense program, of interest in geophysics and ballistics, this monograph may prove useful in putting some special investigations, which are capable of application in these fields, in a more available form.

MICHAEL GOLOMB

Introduction to Abstract Algebra. By C. C. MacDuffee. New York, Wiley, 1940. 7+303 pp. $4.00.

The rapid advances in algebra within the last few years have been largely due to an exploitation of the powerful methods of abstract algebra. Accordingly there has been a tremendous increase in the interest in this subject, so that most colleges and universities offer at least one course in abstract algebra. However, students with only an undergraduate course in the theory of equations as a background in algebra frequently have considerable difficulty with the available texts—not so much in reading the proofs as in grasping the significance of the abstract theories. The present book was written primarily as a text for beginning graduate students and is designed to fill the gap between the usual text in the theory of equations and the more advanced texts on algebra. It should prove to be a valuable addition to the growing list of texts on abstract algebra.

The material is organized in such a way that concrete instances of

a theory are always at hand before the abstract formulation is made. How this is done, as well as the general scope of the book, will be indicated by brief summaries of the different chapters.

Chapter I. The Theory of Numbers. The book begins with a statement of Peano's postulates for the positive integers, followed by a rapid derivation of the simplest properties of the positive integers and an introduction of zero and the negative integers in terms of pairs of positive integers. Incidentally, rational numbers are not presented from this abstract point of view until Chapter V, although it is assumed in Chapters III and IV that the student has an intuitive knowledge of the rationals. The remainder of the chapter is devoted to a study of such topics as the fundamental theorem of arithmetic, congruences in one or more unknowns, residue classes, theorems of Fermat and Euler, primitive roots, quadratic residues and the law of quadratic reciprocity.

Chapter II. Finite Groups. After the definition of a *group* and derivations of a few properties of groups in general, the rest of the chapter deals only with finite groups. The set of automorphisms of a mathematical system is used as an important illustration of a group. A body of theorems is established, culminating in the Jordan-Hölder Theorem, which is essential for the Galois theory of the succeeding chapter. Direct products are defined and discussed briefly. Then follows the introduction of permutation groups and a proof that every finite group can be represented as a regular permutation group. The notions of transitivity and primitivity are not mentioned. The chapter closes with a derivation of the principal theorems about finite Abelian groups—existence of a basis, invariants, and related topics.

Chapter III. Algebraic Fields. The intuitive point of view is used to a considerable extent in this chapter, a more abstract treatment of fields being postponed until later. Quadratic and cubic fields are studied in some detail before more general algebraic fields are introduced. The Galois group of an equation is introduced as a group of automorphisms of its root field, and is then shown to be isomorphic to a group of permutations of the roots of the given equation. There follows a brief discussion of the Galois theory, including a derivation of Galois' criterion for solvability by means of radicals.

Chapter IV. Integral Algebraic Domains. Quadratic integral domains are studied in detail as a background for domains of higher degree. Unique factorization into primes is shown to exist in some quadratic domains but not in others, thus leading to the concept of *ideal* and the unique factorization of ideals into prime ideals. The

actual determination of all prime ideals in a quadratic domain is carried out. Next the integral domain consisting of all algebraic integers in an algebraic field of degree n is considered, and ideals are introduced. The arithmetic of ideals is again carried far enough to obtain the unique factorization theorem.

Chapter V. Rings and Fields. This chapter opens with abstract definitions of *ring*, *non-commutative field*, and *integral domain*. Incidentally, the reviewer is a bit unhappy at the use of a terminology according to which a commutative field is a special case of a non-commutative field. A *principal ideal ring* is defined as an integral domain having certain additional properties. Later on, when ideals in general rings have been introduced, it is shown that a principal ideal ring is an integral domain in which every ideal is principal, thus justifying the earlier terminology. The abstract construction of the quotient field of an integral domain is carried through in detail. Then follows the definition of a *prime field* and the determination of all prime fields. Polynominal rings are introduced abstractly, and various questions of factorization are discussed. Ideals are defined, and the notion of *residue class ring* is introduced, thus leading to the determination of all rings homomorphic to a given ring. A satisfactory treatment of algebraic extensions of a given field can now be carried through without use of the fundamental theorem of algebra. Galois (finite) fields are introduced and fundamental results, including a determination of the group of automorphisms, are derived. The concluding section considers the field $F(\lambda)$ obtained from F by a transcendental extension and, in particular, characterizes all automorphisms of $F(\lambda)$ leaving elements of F fixed.

Chapter VI. Perfect Fields. If a and b are elements of a field F, a function $\phi(a)$ is a *valuation* for F if $\phi(a)$ is a positive number or 0 of some ordered field, subject to the further conditions:

$$\phi(a) > 0 \text{ for } a \neq 0,\ \phi(0) = 0,\ \phi(ab) = \phi(a)\phi(b),\ \phi(a+b) \leqq \phi(a) + \phi(b).$$

The real field is obtained as the field of regular sequences of rational numbers, with the ordinary absolute value as the valuation used in defining regularity. This field is then shown to be *perfect* with respect to this valuation, that is, it is incapable of further extension by means of regular sequences of real numbers. The field of complex numbers is obtained by a simple algebraic extension of the real field, and a proof is given that every algebraic equation with complex coefficients has a complex root. Another valuation of the rational field is obtained as follows. Let p be a fixed rational prime. If $a \neq 0$ is a rational number, it is uniquely expressible in the form $(r/s)p^n$, where

r and s are integers prime to p. Then $\phi(0)=0$, $\phi(a)=p^{-n}$ is a valuation of the rational field. The Hensel p-adic number field Ωp is obtained by use of this valuation in the same way in which the reals are obtained from the rationals by use of the absolute value. There follows a series representation of numbers of Ωp and a characterization of the rational numbers in Ωp as those elements with periodic series. It is also shown that a p-adic extension of an algebraic field can always be obtained with a prime ideal playing a rôle analogous to that of the prime p in the case of the rational field.

Chapter VII. Matrices. After a preliminary study of linear systems, the author introduces the concepts of *array* and *determinant* of a square array. A *matrix* is defined as an element of a total matrix algebra. After some preliminary results, the Hamilton-Cayley theorem is obtained, and also a characterization of the minimum equation of a matrix. Some ten pages are devoted to a study of matrices over a principal ideal ring, including such topics as greatest common right (left) divisor, elementary divisors and normal form. The familiar rational canonical form of a matrix over a field is then obtained, as well as the Jordan form for a matrix over the complex field. The regular representations, by matrices, of a finite algebraic field $F(\rho)$ over F are given as an application of the theory of matrices, and probably also as an introduction to the more general representation theory of the next chapter. After a discussion of polynomials in a single matrix, the chapter concludes with a brief treatment of *direct products* of matrices.

Chapter VIII. Linear Associative Algebras. As a concrete introduction to the general theory, the author begins with an unusually detailed study of quaternions, including some new results, due to C. G. Latimer, on quaternion ideals and factorization. Then follows a brief discussion of division algebras, generalized quaternions and cyclic algebras. Linear algebras with unit element are presented in some detail before those without unit element. Further topics treated briefly are regular representations, direct sums, direct products, ideals and the radical of an algebra. General structure theorems are not proved although the principal ones are stated at the end of the chapter.

There are numerous exercises, several sets in each chapter, which give the student an opportunity to test his grasp of the subject, as well as furnishing additional concrete illustrations of the general theories. The author states that students should work all the problems, as they are designed to supplement the rest of the book in an essential way. At the end of each chapter there occurs a short and

well-chosen list of suggested texts for further reading. At the end of the book there are tables of Greek and German letters, and a list of some of the more important symbols used in the text.

On the whole the book is well written and the typography excellent. The reviewer noticed a few misprints and even an occasional slip. However, these are mostly of a minor nature and should cause little or no confusion to the student. The book is a valuable and timely addition to the available texts on algebra.

NEAL H. MCCOY

Gap and Density Theorems. By Norman Levinson. (American Mathematical Society Colloquium Publications, vol. 26.) New York, American Mathematical Society, 1940. 8+246 pp. $4.00.

In this book the author confines himself to a detailed study of a few salient topics in gap and density theory; he does not attempt to write a systematic treatise on the subject. The book is in form essentially a collection of research papers; it achieves unity principally through the author's repeated application of similar methods to a variety of problems. Most of the contributions to gap and density theory contained in the book are the author's own work, some of the most remarkable of them being published here for the first time. The principal topics treated are, on the one hand, the influence of the distribution of a sequence of numbers $\{\lambda_n\}$ on the closure properties of the sequence $\{e^{i\lambda_n z}\}$, and the closely related topic of the influence of the distribution of the λ_n on the growth of analytic functions which vanish or are otherwise restricted at points $z=\lambda_n$; and, on the other hand, general Tauberian theorems involving gap conditions. Among the topics not treated are, for example, the Paley-Wiener theory of "pseudoperiodic" functions, and Bochner's generalizations of it. The extensive "classical" theory connecting gap or density properties of a sequence $\{\lambda_n\}$ with the position of the singularities of the function having the Dirichlet series $\sum a_n e^{-\lambda_n z}$ is represented by one theorem. The author expects his readers to be familiar with approximately the amount of information contained in Titchmarsh's *Theory of Functions*. Familiarity with the Colloquium Publication of Paley and Wiener is not prerequisite, but would be advantageous for a reader. The author collects in an appendix the auxiliary theorems which he most frequently uses. His principal tools are such things as Jensen's theorem, Carleman's theorem which is its analogue for a half-plane, Phragmén-Lindelöf theorems, and the L^2 theory of Fourier transforms; these he combines in ingenious and often unexpected ways.

Many of the proofs are long and involve elaborate calculations. However, the author has been helpful in many places by prefacing his longer proofs with applications of his methods to simpler problems, so that the reader can see the essential ideas unobscured by details. At times, on the other hand, he demands considerable background from the reader; for example, the "obvious" statement at the beginning of Chapter V seems to be obvious principally because it is a known true theorem.

On many of the problems which he discusses, the author says the last word, since the problems are solved by theorems which are shown by examples to be "best possible." In most cases he generalizes existing theorems by constructing entirely new proofs, rather than by generalizing existing proofs; his methods have allowed him to attain wide generality and great precision. Anyone who is interested in Fourier integrals, the growth of analytic functions, or Tauberian theorems, should profit from studying the author's methods of attack. There is no reason for supposing that their power is yet exhausted.

A chapter-by-chapter résumé follows. In Chapters I and II the central problem is the closure of a sequence $\{e^{i\lambda_n x}\}$. Such a sequence is said to be closed[1] L^p if $\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\,dx=0$ for all n, and $f(x)\in L^p$, imply $f(x)=0$ almost everywhere. As Szász has pointed out, studying the closure of a sequence $\{e^{i\lambda_n x}\}$ amounts to studying the zeros of entire functions of the form $\int_{-\pi}^{\pi} f(t)e^{ixt}dt$; and as Paley and Wiener observed, closure theorems have as corollaries theorems stating that a function with gaps in its sequence of Fourier coefficients cannot vanish over too long an interval without vanishing identically. The first two chapters contain a variety of closure and gap theorems, some in sharper form than when they were previously published. The few that are not due to the author in their present degree of generality are due to Miss Cartwright. Chapter III contains the theorems connecting the distribution of the zeros of an entire function of exponential type with its rate of growth along the real axis, used in proving the closure theorems of Chapters I and II. The essential idea is that if the function does not grow too fast along the real axis most of its zeros must lie near the real axis, and the sequence of zeros has a density in the right and left half-planes; the author's theorems are more precise than earlier results of this character.

[1] The author follows Paley and Wiener, who interchange the customary terms. Most authors would write "complete in $L^{p'}$," where $p'=p/(p-1)$, instead of "closed L^p." Since the properties of closure in the usual sense and in the author's are equivalent, no great harm is done.

Chapter IV is devoted to non-harmonic Fourier series. If a sequence $\{e^{i\lambda_n x}\}$ is closed L^p, every function of L^p can be represented as the limit (in the topology of L^p) of a sequence of linear combinations of the functions $e^{i\lambda_n x}$. It was observed by Paley and Wiener that the restriction $|\lambda_n - n| \leqq c$, with sufficiently small c, will ensure in additon that the sequence $\{e^{i\lambda_n x}\}$ will serve to expand functions into series preserving many of the properties of ordinary Fourier series; such series they called non-harmonic Fourier series. Paley and Wiener showed that there is an L^2 theory if $c < 1/\pi^2$. Levinson showed, by different methods, that there is an L^p theory $(1 < p \leqq 2)$ if $c < (p-1)/(2p)$, but not necessarily if $c \leqq (p-1)/(2p)$. His results are reproduced in Chapter IV, together with an interesting inequality of Hardy and Levinson which is connected with the problem.

Chapter V contains various developments of the author's results which state (roughly) that if a function is "small at infinity" and its Fourier transform coincides with an analytic function over an interval, then its Fourier transform continues to coincide with the analytic function over the largest interval of the real axis on which that function continues to be analytic. Such results generalize gap theorems stating that if the Fourier transform vanishes over an interval it vanishes identically (the analytic function being in this case identically zero). Chapter VI contains very precise estimates for the rate of growth of an entire function of exponential type whose zeros have a density; these are applied to prove the classical theorem of Pólya connecting the density of the exponents of a Dirichlet series with the position of its singular points on its abscissa of convergence. In Chapter VII the estimates are combined with results from Chapter V to establish conditions under which the rate of growth of an analytic function on a sequence of points on or near a line determines its rate of growth along the whole line. Theorems of this character were obtained by V. Bernstein from the theory of Dirichlet series; the author proves Bernstein's theorems by simpler and more natural methods, and then develops a method of proof in which the results of Chapter V can be used to make the results much more precise.

The greater part of the material of Chapters I–VII has been published before; Chapters VIII and IX are new material. Pólya set as a problem the theorem that an entire function of order one and zero type is constant if it is bounded at the integers. The author replaces the integers by a sequence $\{\lambda_n\}$ sufficiently near the integers; in particular, if

$$\lambda_n - n = O\{n/(\log n)^{1+\delta}\}$$

as $n\to\infty$, for some positive δ, the theorem remains true; the theorem of Chapter VII is still more general, but too complicated to reproduce here. Remarkably enough, the condition above is approximately "best possible," since the author shows that there are sequences $\{\lambda_n\}$ satisfying the condition $\lambda_n - n = O(n/\log n)$, but such that there exist entire functions of zero exponential type bounded on the sequence and not constant; thus the existence of a positive density for the sequence $\{\lambda_n\}$ is not enough to make the theorem true. The proof of the theorem is based on a powerful inequality for analytic functions. Let $M(x)$ be a positive even function, decreasing for increasing $|x|$, and let $M(x)\to\infty$ as $x\to 0$. Let $f(z)$ be analytic in the rectangle $|x|\leqq a$, $|y|\leqq b$, and let $|f(x+iy)|\leqq M(x)$ for $x+iy$ in the rectangle. If $\int_0^1 \log\log M(x)\,dx$ converges, then $|f(x+iy)|\leqq C$ for $|x|\leqq a$ and $|y|\leqq b(1-\delta)$, where C depends only on δ and on $M(x)$. This theorem seems likely to prove useful in many problems other than that to which it is applied in this chapter. Chapter VIII is devoted to showing that the results of Chapter VII (both the theorem just quoted and the results on functions of zero type) are best possible. This is done by constructing a suitable function of zero type by means of its representation as an infinite product; this is by no means a simple task (it requires 33 pages).

Chapters X, XI, and XII are devoted to gap Tauberian theorems; Chapter X is substantially the same as a recent paper of Levinson's, while the other two chapters consist of new material. A Tauberian theorem for Dirichlet series states that if $f(x)=\sum_1^\infty a_k e^{-\mu_k x}$ for $x>0$ and $f(x)\to s$ as $x\to 0+$, then under suitable conditions $\sum a_k = s$. Most such theorems involve restrictions on the size of the a_k; but a classical theorem of Hardy and Littlewood states that if $\mu_{k+1}/\mu_k \geqq \theta > 1$, no restriction on the a_k is necessary. In the language of Wiener's general Tauberian theorems, one has to consider when, from

$$f(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\,dy, \qquad \int_{-\infty}^\infty K(y)\,dy = 1,$$

and $f(x)\to s$ as $x\to\infty$, one can deduce that $\sum a_n = s$. [The Dirichlet series case corresponds to $K(y)=e^y \exp(-e^y)$.] In Wiener's theory the a_n are restricted in size and $K(y)$ is required to have a Fourier transform which does not vanish anywhere. For a gap Tauberian theorem, one might expect to be able to omit restrictions on the a_n provided that $\lambda_{n+1}-\lambda_n \geqq L > 0$. The author shows by an example that there are kernels $K(y)$ [e.g., $e^{-y^2/2}$] whose Fourier transforms do not vanish but for which such a Tauberian theorem does not hold. In Chapter X

he shows that there is a gap Tauberian theorem for $K(y)$ provided that its Fourier transform satisfies sufficiently strong additional conditions. His proof, like Wiener's proof of the Hardy-Littlewood theorem, depends on reducing the Tauberian theorem to a lemma asserting the existence of a suitable set of biorthogonal functions. While in Wiener's proof this set was relatively easy to construct, in the general case it is quite difficult.

In Chapter XI it is shown that for more general kernels than those satisfying the conditions of Chapter X there are still Tauberian theorems in which the size of the a_n is not restricted, provided that the gaps in the sequence $\{\lambda_n\}$ are large enough; for the kernel $e^{-y^2/2}$ the condition is essentially $\sum 1/\lambda_n = \infty$. In Chapter XII it is shown that for this kernel the original condition $\lambda_{n+1}-\lambda_n \geqq L>0$ still ensures the existence of a Tauberian theorem if the a_n are restricted in size (though only rather mildly compared with the restrictions required without the gap condition on the λ_n). It is also indicated how the method used would apply to more general kernels.

R. P. Boas, Jr.

NOTES

A summer session of special interest is being held at Brown University this year. It is devoted to advanced instruction and research in mechanics, a program designed for the purpose of increasing the effectiveness of research in essential American industries. The summer course is approved as a part of the Engineering Defense Training Program of the U. S. Office of Education. The Carnegie Corporation of New York has made a grant toward its support. Similar sessions are planned for the period October 1, 1941–January 31, 1942, and February 11–June 6, 1942. Those who wish information about these sessions should communicate with the Dean of the Graduate School, Brown University, Providence, Rhode Island.

Among the members of the U. S. National Defense Research Committee who visited Great Britain were Dr. Warren Weaver and Professor J. L. Tate.

Professor Sydney Chapman of Imperial College, South Kensington, has been elected president of the Royal Astronomical Society. Professor H. C. Plummer of the Military College of Science, Woolwich, has been elected vice president.

Professor L. W. Cohen of the University of Kentucky has been named president of the university branch of the American Association of University Professors.

Dean L. P. Eisenhart of Princeton University has been appointed one of the directors of the National Science Fund of the National Academy of Sciences.

Professor W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America, has announced that the Department of Mathematics of Brooklyn College, Brooklyn, New York, has won the first prize of $500 in the fourth annual William Lowell Putnam Mathematical Competition, held March 1, 1941. The members of the team were Richard Bellman, Peter Chiarulli, Hyman Zimmerberg. The second prize of $300 is awarded to the Department of Mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania, the members of whose team were S. I. Askovitz, Hyman Kamel, P. C. Rosenbloom. The third prize of $200 is awarded to the Department of Mathematics of the Massachusetts Institute of Technology, Cambridge, Massachusetts, with team members J. R. R.

Baumberger, Eugene Calabi, W. S. Loud. In addition to these prizes to the departments of mathematics with winning teams, a prize of $50 each is awarded to the following five persons whose scores ranked highest in the six-hour examination (names listed alphabetically): R. F. Arens, University of California at Los Angeles; S. I. Askovitz, University of Pennsylvania; A. M. Gleason, Yale University; E. L. Kaplan, Carnegie Institute of Technology; P. C. Rosenbloom, University of Pennsylvania. Of these five, one will later be chosen to receive a $1000 year scholarship at Harvard University, this award to be announced later. The members of the three winning teams will receive individual cash awards according to the ranks of their teams, and all individuals receiving awards will also receive medals. Honorable mention has been awarded this year to three teams and to six individuals. The teams are from the Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, members of the team being R. E. Beatty, E. L. Kaplan, N. H. Painter; the Department of Mathematics, Cooper Union Institute of Technology, New York, members of the team being Murray Klamkin, Benjamin Lax, Samuel Manson; and the Department of Mathematics, Yale University, New Haven, the team members being A. M. Gleason, G. R. MacLane, D. M. Merrill. The five individuals receiving honorable mention are Richard Bellman, Brooklyn College; Harvey Cohn, College of the City of New York; W. S. Loud, Massachusetts Institute of Technology; G. R. MacLane, Yale University; Samuel Manson, Cooper Union Institute of Technology; Hyman Zimmerberg, Brooklyn College.

Dr. C. A. Coulson of University College, Dundee, has been elected a fellow of the Royal Society of Edinburgh.

Professor T. G. Room of the University of Sydney has been elected to fellowship in the Royal Society, London.

Professor G. A. Bliss of the University of Chicago will retire on September 30 with the title of professor emeritus.

Professor G. C. Evans of the University of California and Professor J. R. Kline of the University of Pennsylvania have been elected to membership in the American Philosophical Society.

Professor E. V. Huntington of Harvard University has been given the title of professor emeritus.

Dr. Joseph Slepian of the Westinghouse Electric and Manufacturing Company and Professor T. Y. Thomas of the University of

California at Los Angeles have been elected to membership in the National Academy of Sciences.

Professor H. S. Uhler of Yale University has been given the title of professor emeritus.

Professor Mabel M. Young of Wellesley College has been given the title of professor emeritus.

Dr. Theodor Esterman and Mr. A. C. Stevenson have been appointed readers in mathematics at University College, London.

Associate Professor A. A. Albert of the University of Chicago has been promoted to a professorship.

Associate Professor W. L. Ayres of the University of Michigan has been appointed head of the department of mathematics at Purdue University.

Mr. J. J. Barron of St. Joseph College, Hartford, Connecticut, has been promoted to an associate professorship.

Associate Professor M. A. Basoco of the University of Nebraska has been promoted to a professorship.

Dr. C. B. Boyer of Brooklyn College has been promoted to an assistant professorship.

Assistant Professor Richard Brauer of the University of Toronto has been on leave of absence at the University of Wisconsin as visiting lecturer during the past semester.

Assistant Professor B. L. Brown of Amherst College has been promoted to an associate professorship.

Assistant Professor O. E. Brown of Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor I. S. Carroll of Syracuse University has been promoted to an associate professorship.

Dr. A. H. Clifford of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. S. H. Gould of Victoria College, Toronto, has been promoted to an assistant professorship in classics.

Dr. Harriet M. Griffin of Brooklyn College has been promoted to an assistant professorship.

Assistant Professor D. C. Lewis of the University of New Hampshire has been promoted to an associate professorship.

Assistant Professor Saunders MacLane of Harvard University has been promoted to an associate professorship.

Associate Professor C. J. Rees of the University of Delaware has been promoted to a professorship.

Dr. R. F. Rinehart of Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor J. H. Roberts of Duke University has been promoted to an associate professorship.

Assistant Professor P. J. Rulon of Harvard University has been promoted to an associate professorship.

Assistant Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to an associate professorship.

Associate Professor G. W. Starcher of Ohio University has been promoted to a professorship.

Dr. C. C. Torrance of Case School of Applied Science has been promoted to an assistant professorship.

Dr. J. W. Tukey of Princeton University has been promoted to an assistant professorship.

Mr. R. L. Wilson of the University of Wisconsin has been appointed to an assistant professorship in military science and tactics at the Alabama Polytechnic Institute.

Associate Professor H. N. Wright of the College of the City of New York has been appointed acting president of the College in succession to Dr. N. P. Mead.

Dr. G. W. Whaples of Indiana University has been made an assistant in mathematics at the Institute for Advanced Study.

The following appointments to instructorships are announced: Cornell University: Dr. E. H. Hadlock; Fenn College, Cleveland, Ohio: Mr. D. H. Erkiletian; Harvard University: Mr. L. H. Loomis; The Hill School, Pottstown, Pennsylvania: Mr. W. N. Huff; Johns Hopkins University: Mr. E. A. Coddington; University of Kansas City: Mr. D. E. Kibbey; University of Maryland: Dr. Martha H. Williams; Northwestern University: Dr. J. W. Givens; University of

Pennsylvania: Dr. A. D. Wallace, Mr. P. M. Whitman; Yale University: Mr. J. G. Herriot.

In Nature has appeared an announcement of the death of Dr. E. L. Ince, lecturer on technical mathematics at the University of Edinburgh, on March 16, 1941, at the age of forty-nine years.

Assistant Professor G. M. Hayes of the College of the City of New York died January 2, 1941.

Dr. Ethel I. Moody of Pennsylvania State College was killed in an automobile accident April 11, 1941.

Professor Emeritus J. E. Trevor of Cornell University died May 4 1941, at the age of seventy-six years. He had been a member of the American Mathematical Society since 1921.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

293. A. H. Clifford: *Matrix representations of completely simple semigroups.*

By a representation $\mathfrak{T}^*$ of degree n of a semigroup S is meant a mapping $a \to T^*(a)$ of S into the set of all square matrices of n rows and columns such that $T^*(ab) = T^*(a)T^*(b)$. A. Suschkewitsch (Communications de la Société Mathématique de Kharkow, vol. 6 (1933), pp. 27–38) partially solved the problem of determining all representations of a type of semigroup which he calls a "Kerngruppe." In the terminology of D. Rees (Proceedings of the Cambridge Philosophical Society, vol. 36 (1940), pp. 387–400), the latter is a "completely simple semigroup without zero." The present paper completes the solution, at the same time allowing S to be any completely simple semigroup. To any such S belongs a "structure group" G, and every representation $\mathfrak{T}^*$ of S is an "extension" of a representation $\mathfrak{T}$ of G. To any given $\mathfrak{T}$ corresponds a certain matrix H, possibly infinite. $\mathfrak{T}$ admits an extension $\mathfrak{T}^*$ of finite degree if and only if H has finite rank h. If $\mathfrak{T}$ has degree n, there exists a "basic" extension $\mathfrak{T}_0^*$ of degree $n+h$, and every extension $\mathfrak{T}^*$ of $\mathfrak{T}$, even though it may be indecomposable, reduces into $\mathfrak{T}_0^*$ and null representations. (Received March 31, 1941.)

294. C. J. Everett: *Vector spaces over rings.*

Every basis of a vector space V_n (with n basis elements) over a ring K with right ideal maximal condition has n elements. A ring is exhibited over which any V_n is always a V_1. Every submodul M of a V_n over ring K has a basis of $b(M) \leq n$ elements if and only if (property P) every right ideal of K is principal of type rK, where r is a non-left-zero-divisor. For V_n over K with property P, $b(M)$ is a positive modular functional (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 40). If K has property P and is without zero-divisors, the metric homomorph (loc. cit., Theorem 3.10) of the lattice of submoduls of V_n over K is lattice-isomorphic with the lattice of submoduls of V_n over the quotient field of K. For V_n over K of type P, $b(M)$ is sharply positive (loc. cit., p. 41) if and only if K is a quasi-field. (Received May 26, 1941.)

295. A. W. Jones: *On the characterization of groups whose lattices satisfy specified lattice identities.*

It is shown that the lattice of a group is simple, complemented, and modular (that is, capable of abstractly representing a finite projective geometry) if and only if the group is either: (1) abelian of prime power order and with prime order elements

only, or (2) a non-abelian group of order pq^n generated by the elements a and b_i $(i=1, 2, \cdots, n)$ which satisfy the relations $a^p=b_i^q=1$ and $a^{-1}b_ia=b_i^u$ (p divides $q-1$ and q divides $u^p-1\neq 0$). A group lattice is complemented and modular if and only if the group is the direct product of groups of relatively prime orders and of types (1) or (2) above. (Received May 21, 1941.)

296. Leonard Carlitz: *An analogue of the Bernoulli polynomials.*

The polynomials in question are defined by means of $\psi(tu)/u\psi(t)=\sum\beta_m(u)t^m/g_m$, which may be compared with $t/\psi(t)=\sum B_mt^m/g_m$. (For the quantities involved see *An analog of the Staudt-Clausen theorem*, Duke Mathematical Journal, vol. 3 (1937), p. 503; also, vol. 7 (1940), p. 62.) If u is put equal to U, a polynomial in x over $GF(p^n)$, then $\beta_m(U)$ becomes a rational fraction in x alone. This paper contains various theorems of an arithmetic nature on $\beta_m(U)$, in particular a theorem of the Staudt-Clausen type on the fractional part. (Received April 11, 1941.)

297. Mark Kac: *Remark on the distribution of values of the arithmetic function $d(m)$.*

Let $d(m)$ denote the number of different divisors of m (1 and m included) and $r_n(\omega)$ the number of positive integers less than or equal to n for which $d(m)\leqq 2^{[\log\log n+\omega(2\log\log n)^{1/2}]}$. Then $\lim_{n\to\infty}r_n(\omega)/n=\pi^{-1/2}\int_{-\omega}^{\omega}e^{-u^2}du$. (Received May 2, 1941.)

298. Joseph Lehner: *A partition function connected with the modulus five.*

Let $p_1(n)$, $p_2(n)$ denote the number of partitions of a positive integer n into summands of the form $5l\pm 1$ and $5l\pm 2$, respectively. These functions occur in identities of I. Schur, in which they are related to the number of partitions of n into summands which differ by at least two. Convergent series for $p_1(n)$ and $p_2(n)$ are derived by using the Hardy-Littlewood method in the strengthened form of Kloosterman-Rademacher. The transformation equations of the generating functions of $p_1(n)$, $p_2(n)$, which are closely related to certain modular forms of dimension zero, yield new relations in the theory of theta-functions. The familiar exponential sums which arise from the application of the Hardy-Littlewood method are reduced by a number-theoretic method to Kloosterman sums, thereby permitting the sharper estimate $O(n^{1/2}k^{2/3+\epsilon})$ which is needed for the application of Rademacher's method to modular forms of dimension zero. (Received April 2, 1941.)

299. J. B. Rosser: *A generalization of the euclidean algorithm to several dimensions.*

If $Y=\sum a_iX_i$, the a's being integers, Y is said to be an I.L.C. (integral linear combination) of the X's. A set of vectors $U_1, \cdots, U_r$ is called a G.C.F. of a set of vectors $V_1, \cdots, V_s$ if: I. The U's are linearly independent. II. Each U is an I.L.C. of the V's. III. Each V is an I.L.C. of the U's. A set of V's is said to be commensurable if the set of I.L.C.'s of the V's has no limit point. It is proved that every commensurable set of vectors has a G.C.F. The G.C.F. of two commensurable collinear vectors is a single vector, which can be found by the euclidean algorithm. Generalizing the euclidean algorithm in a rather obvious fashion, one has a procedure for finding a G.C.F. of more general sets of vectors. This algorithm solves numerous problems

relative to positive definite quadratic forms and Diophantine equations. For two incommensurable collinear vectors, the euclidean algorithm becomes the continued fraction algorithm, a powerful tool in approximation problems. It is conjectured that the generalized algorithm is as effective for problems of simultaneous approximation as the continued fraction algorithm is for simple approximation problems. To support this conjecture, satisfactory solutions of such problems are obtained by use of the generalized algorithm. (Received April 24, 1941.)

300. Albert Whiteman: *Sums connected with the partition function.*

The sum $A_k(n)=\sum \exp(\pi i s(h,k)-2\pi ihn/k)$, where h runs over a reduced residue system with respect to the modulus k and $s(h,k)$ is a Dedekind sum, appears in Rademacher's formula for the number of partitions of n. On the basis of a different expression for this sum Lehmer (Transactions of this Society, vol. 43 (1938), pp. 271–295) factored the $A_k(n)$ according to the prime number powers contained in k, and evaluated the $A_k(n)$ in the case in which k is a prime or a power of a prime. A new approach to the first of these results has recently been given in a paper by Rademacher and Whiteman (American Journal of Mathematics, vol. 63 (1941), pp. 377–407). In the present paper the second of these results is derived by a method which is considerably simpler than Lehmer's. The paper also contains a new method for evaluating certain generalized Kloosterman sums. (Received May 29, 1941.)

ANALYSIS

301. R. P. Agnew: *On limits of integrals.*

It is shown that existence of $\lim_{A\to\infty}\int_A^{A+\lambda}f(t)dt$, for each λ in some set having positive measure, implies that the limit exists and is uniform over each finite interval of values of λ. The result is applied to prove two theorems of Iyengar (Proceedings of the Cambridge Philosophical Society, vol. 37 (1941), pp. 9–13) and the following Tauberian theorem. If $F(t)$ is absolutely continuous over each finite interval, if $\lim_{A\to\infty}\int_A^{A+\lambda}[F(t)-F'(t)]dt=0$ for each real λ, and if $\lim_{t\to\infty}e^{-t}F(t)=0$, then $\lim_{t\to\infty}F(t)=0$. (Received April 11, 1941.)

302. Stefan Bergman: *A method for summation of series of orthogonal functions of two variables.*

Suppose $\{\Phi_\nu(\phi)\}$ $[(\phi)=(\phi_1,\ \phi_2)]$ is a system of O.N. functions $[0\leqq\phi_k\leqq 2\pi,$ $\Phi_k(\phi_1+2\pi,\ \phi_2)=\Phi_k(\phi_1,\ \phi_2+2\pi)=\Phi_k(\phi_1,\ \phi_2),\ k=1,\ 2]$, $\sigma(e)$ a completely additive set function, $f(\phi_1,\phi_2)\in L^{1+p}(p>0)$, $a_\nu=\int_0^{2\pi}\int_0^{2\pi}\Phi_\nu d\sigma$, $b_\nu=\int_0^{2\pi}\int_0^{2\pi}\Phi_\nu f d\phi_1 d\phi_2$. Consider the series (1) $\sum_{\nu=0}^{\infty}a_\nu\Phi_\nu(\phi_1,\phi_2)$ and (2) $\sum_{\nu=0}^{\infty}b_\nu\Phi_\nu(\phi_1,\phi_2)$. Let $\mathfrak{M}$ be a four-dimensional domain of the type described in Mathematische Annalen, vol. 104 (1931), pp. 611–636, with the distinguished boundary surface $\mathfrak{F}=E[z_k=h_k(\phi_1,\ \phi_2),\ k=1,\ 2]$. Let $\Psi_\nu(z_1,\ z_2)$, $\nu=1,$ $2,\cdots$, be the functions of the extended class which assume the values Φ_ν on $\mathfrak{F}$, and (3) $S(z_1,\ z_2)=\sum_{\nu=0}^{\infty}a_\nu\Psi_\nu(z_1,\ z_2)$, (4) $F(z_1,\ z_2)=\sum_{\nu=0}^{\infty}b_\nu\Psi_\nu(z_1,\ z_2)$. The series (4) converges absolutely and uniformly in every closed subdomain of $\mathfrak{M}$ for $p>1$. Using the results of Jessen, Marcinkiewicz and Zygmund (Fundamenta Mathematicae, vol. 25 (1935), pp. 217–234), Bergman and Marcinkiewicz (Fundamenta Mathematicae, vol. 33 (1939), pp. 75–94) and Bers (Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 1273–1275 and 1475–1477) it is shown that S possesses a finite sectorial limit at the point $\{h_1(\phi),\ h_2(\phi)\}$ if σ possesses a finite strong derivative at (ϕ), and F possesses the sectorial limit f almost everywhere on $\mathfrak{F}$. (Received May 2, 1941.)

303. R. P. Boas: *Generalized Laplace integrals.*

The generalized Laplace integrals discussed have the form $f(z)=\int_0^\infty g(z, t)\phi(t)dt$, where $g(z, t)$ is in some sense "nearly" equal to the kernel e^{-zt}. The conditions imposed on $g(z, t)$ are analogous to those which have been imposed on functions $g_n(z)$ to make the series $\sum a_n g_n(z)$ behave like a power series (cf. Boas, Transactions of this Society, vol. 48 (1940), pp. 467–487). An application is made to the kernel $g(z, t)=(2zt/\pi)^{1/2}K_\nu(z, t)$ recently discussed by Meijer (Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 43 (1940), pp. 599–608, 702–711; see Mathematical Reviews, vol. 2 (1941), p. 96); here $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$, and $K_\nu(z)$ is the usual notation for a Bessel function of imaginary argument; when $\nu=\pm\frac{1}{2}$, $g(z, t)=e^{-zt}$. (Received May 19, 1940.)

304. D. G. Bourgin: *On reflexive Banach spaces.*

Several theorems are obtained generalizing a result of Ghantmacher and Smulian (Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 17 (1937), p. 91). For instance: *If the unit sphere of E is sectionally compact of order $\aleph_\alpha$ in the weak topology of E as elements and if E^* contains a dense set, of power less than or equal to $\aleph_\alpha$, in the sense of the weak topology of E^* as functionals, then E is reflexive.* Some incidental results are also available. Thus a theorem implying the possibility of interchange of order of certain Moore-Smith limiting operations is: *If* (a) $\{f^{(\omega)}\}$ *is a $w^*(E^*)$ dense set of power less than or equal to $\aleph_\alpha$ and* (b) $\{x^\rho\}$, *ρ a directed set of power less than or equal to $\aleph_\alpha$, is contained in a $w(E)$ compact (of order $\aleph_\alpha$) set, Q, and* (c) $f^l(x^\rho)$ *converges in the Moore-Smith sense to $f^l(x_0)$ for each l, then $\{x^\rho\}$ is $w(E)$ convergent to x_0 in the Moore-Smith sense.* (Received April 2, 1941.)

305. J. H. Curtiss: *Riemann sums and the fundamental polynomials of Lagrange interpolation.*

Let C denote an arbitrary Jordan curve of the z-plane and let $z=\phi(w)$ be an analytic function which maps the exterior K of C conformally onto the region $|w|>1$ so that the points at infinity correspond. Let $\phi'(\infty)=c$. Let $\omega_n(z)=\prod_{k=1}^n[z-\phi(e^{2\pi ik/n})]$, $n=1, 2, \cdots$. Then if C is rectifiable, $\lim_{n\to\infty}\omega_n(z)/(-c^n)=1$ uniformly for z on any closed point set of the region interior to C, and $\lim_{n\to\infty}\omega_n(z)/c^n(w^n-1)=1$ uniformly for z on any closed point set in K. Furthermore, the last equation is true uniformly for z in $C+K$ if $\phi'(w)$ is nonvanishing and of bounded variation for $|w|=1$. Auxiliary results used in the proof include an extension of a known result on Riemann sums and a proof that if $\phi'(w)$ is absolutely continuous for $|w|=1$, then $[\phi(w)-\phi(\bar{w})]/[w-\bar{w}]$ is uniformly absolutely continuous in either w or $\bar{w}$ for $|w|=1$. (Received April 18, 1941.)

306. M. H. Heins: *A generalization of the Aumann-Carathéodory "Starrheitssatz."*

Let F denote a Riemann surface in the sense of Weyl-Radó with the property that its fundamental group is non-abelian. Then the identical map $W\equiv w$ of F onto itself is isolated in the family of analytic maps $\{f(w)\}$ of F into itself in the sense that there exists no sequence of maps of the family $\{f_n(w)\}$ $(f_n\not\equiv w,\ n=1, 2, \cdots)$ which converges pointwise to w as $n\to\infty$. The proof is based upon a study of the transforms of $f(w)$ with respect to a uniformizing map by means of Julia's lemma.

This theorem implies the following immediate corollaries: (a) the theorem of Klein and Poincaré concerning the proper discontinuity of the group of (1, 1) conformal maps of F onto itself where F is a surface of the above type; (b) the theorem of Koebe stating that the number of (1, 1) conformal maps of a plane region of finite connectivity greater than two onto itself is finite; (c) the "Starrheitssatz" of Aumann and Carathéodory. (Received April 2, 1941.)

307. Einar Hille: *Notes on linear transformations.* III. *Semi-groups on Lebesgue spaces.*

This paper deals with the properties of semi-groups on $L_1(-\pi, \pi)$ and $L_1(-\infty, \infty)$, essentially those which are multiplicative in the representation by Fourier series, Fourier transforms or Hermitian series. The first two types coincide with the semi-groups commuting with real translations. The discussion requires the solution of the factor problems of type (L, L) for Fourier transforms and for Hermitian series. Conditions are found for the convergence of T_α to T_0 in various topologies and the best degree of approximation is discussed. It is shown that any convex domain, invariant under addition, of the parameter plane can be the domain of existence of an analytic semi-group in $L_1(-\pi, \pi)$ and the rate of growth of $\|T_\alpha\|$ is studied when the domain is $\Re(\alpha)>0$. (Received May 5, 1941.)

308. Walter Leighton and W. J. Thron: *Convergence criteria for continued fractions.*

In the $z=x+iy$ plane let alternate coordinate axes x', y' be obtained by rotating the x, y axes through an arbitrary angle $0\leq\beta<\pi/2$. Let $R(\beta)$ be the open convex region containing $z=0$ whose boundary is the broken line contour consisting of the two arcs of the parabola $y'^2=\cos\beta\cdot(x'+\frac{1}{4}\cos\beta)$ from infinity to the points of tangency of this parabola with the tangent rays drawn from the point $z=-\frac{1}{4}$ and completed by the segments of these tangent rays extending from $z=-\frac{1}{4}$ to the points of tangency. If the a_n are any complex numbers lying in any closed bounded region in $R(\beta)$, the continued fraction $1+K(a_n/1)$ converges. Further, for $\beta=\pi/2$ let R_1 be the open region defined by $-\frac{1}{4}<\Re(z)<0$, $\Im(z)>0$. If the a_n lie in a bounded closed region in R_1, the given continued fraction converges. In this theorem R_1 may be replaced by R_2: $-\frac{1}{4}<\Re(z)<0$, $\Im(z)<0$. When $\beta=0$ the region described becomes the parabola of Wall and Scott. (Received May 9, 1941.)

309. E. J. McShane: *Sufficient conditions for a weak relative minimum in the problem of Bolza.*

In the problem of Bolza there remains one large gap between the necessary conditions for a minimum and the sufficient conditions. For anormal problems, the condition on the positiveness of the second variation which figures in the sufficient conditions can not be established among the necessary conditions. Here a substitute is proposed for the condition of positiveness of the second variation which is equivalent for normal problems but weaker in anormal problems. It is shown that with this weakened assumption the sufficiency theorem for weak relative minima can still be established. It is conjectured that the new condition (with positiveness replaced by non-negativeness) can be shown to be necessary for a minimum. (Received April 3, 1941.)

310. J. D. Mancill: *On the solutions of a certain class of partial differential equations.*

It is shown in this note that the most general solution of the partial differential equation $(\sum_{i=1}^{n} x_i \partial/\partial x_i)^k F(x) = mF(x)$, $i = 1, 2, \cdots, n$, where k is a positive integer and m is any constant, is the sum of p functions $F_j(x)$, $j = 1, 2, \cdots, p$, $p \leqq k$, homogeneous of degree r_j but otherwise arbitrary; the r_j being the distinct roots of the equation $r(r-1)(r-2)\cdots(r-k+1) = m$. (Received May 19, 1941.)

311. E. N. Oberg: *Notes on the approximation of a function by sums of orthonormal functions.* Preliminary report.

Let $\phi_0, \phi_1, \phi_2, \cdots, \phi_n$ be a set of orthonormal functions on an interval (a, b), each having a derivative at every point. If $S_n(x) = a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n$ is an arbitrary sum of the ϕ's, it is shown that a bound for $|(d/dx)S_n(x)|$ exists in terms of $S_n(x)$. A study is made of the degree of convergence of sums of the form $S_n(x)$ to a function $f(x)$ on (a, b). Lastly, if $\Phi_n(x)$ is a sum of the ϕ's that minimizes the integral of $|f(x) - \Phi(x)|^p dx$, $p > 0$, then under certain hypotheses on $f(x)$, $\Phi_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$. (Received April 7, 1941.)

312. George Polya and Norbert Wiener: *On the oscillation of the derivatives of a periodic function.*

Let $f(x)$ be periodic, real valued and infinitely derivable. Denote by N_k the number of changes of sign of the kth derivative of $f(x)$, in a period. Let $k \to \infty$. *If N_k is bounded, $f(x)$ is a trigonometric polynomial. If $N_k = o(k^{1/2})$, $f(x)$ is an entire function. If $N_k = O(k^{(1-\alpha)/2})$, where $0 < \alpha < 1$, $f(x)$ is an entire function of finite order not exceeding* $(1+\alpha^{-1})/2$. (Received April 19, 1941.)

313. W. T. Reid: *Green's lemma and related results.*

This paper is concerned with a proof of Green's lemma for the general case of a region R which is the interior of a simply closed rectifiable curve J, and where the continuity and differentiability conditions imposed on the functions involved are merely on $R+J$ and R, respectively. There is also established a general form of Cauchy's theorem for a function $f(z)$ which is holomorphic on R, and merely continuous on $R+J$. Finally, there is considered an associated result which is useful in the derivation of edge conditions for double integral problems of the calculus of variations. (Received April 11, 1941.)

314. M. S. Robertson: *The partial sums of multivalently star-like functions.*

Let $f(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n + \cdots$ be regular and multivalently star-like of order p with respect to the unit circle $|z| < 1$. This means that within the unit circle $f(z)$ assumes no value more than p times, at least one value exactly p times, and in addition is star-like, that is, $\Re[zf'(z)/f(z)] > 0$, $|z| < 1$. When $p = 1$, $f(z)$ is univalently star-like. The author shows that for $n > n_0(p)$ the nth partial sum $f_n(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n$ is also multivalently star-like of order p for $|z| < 1 - A(p) \log n/n$ where $2p+1 \leqq A(p) \leqq 2p+2$. The order $\log n/n$ is best possible since the nth partial sum of the power series for $z^p(1-z)^{-2p}$ is multivalently star-like of order p in a circle about the origin whose radius is at most R_n where $R_n \geqq 1 - (2p+1) \log n/n$ and $\limsup_{n\to\infty}(1 - R_n/n^{-1} \log n) = 2p+1$. (Received April 3, 1941.)

315. L. B. Robinson: *Some complete systems of semitensors.*

The first problem is to find a complete system of semitensors of order one associated with the system of linear homogeneous differential equations (A) $y_i'' + \sum_{j=1}^{3} p_{ij}y_j' + \sum_{j=1}^{3} q_{ij}y_j = 0$ $(i = 1, 2, 3)$. The semitensors are transformed thus: $\bar{I} = \Delta_{1i}I_1 + \Delta_{2i}I_2 + \Delta_{3i}I_3$ where the Δ_{ji} are minors of the determinant of the transformation. Write $V_i \equiv D_{i1}I_1 + D_{i2}I_2 + D_{i3}I_3 = f_i$, in which the D_{ij} are minors of the determinant $|Z\ Y\ y|$ and the f_i arbitrary functions of the covariants of A. Solve with respect to the I_j and the complete system of semitensors results. To get semitensors of the second order, write $W_{ij} \equiv V_i \cdot V_j = f_{ij}$, wherein f_{ij} is an arbitrary function of the covariants, assume $I_iI_j \equiv I_{ij} \equiv I_{ji}$, solve with respect to I_{ij} and the complete system of semitensors is obtained. (Received April 7, 1941.)

316. W. J. Trjitzinsky: *Singular Lebesgue-Stieltjes integral equations.*

Integral equations, involving Stieltjes integrals, have been studied by a number of authors and, quite extensively, by N. Gunther. The kernels considered by Gunther are sufficiently "regular" to secure results resembling those of Fredholm and in the case of "symmetry," suitable defined, resembling those of Schmidt. The present work develops comprehensive theories of several types of integral equations, involving "symmetric" kernels essentially more general than those of Gunther; the kernels of the present work are representable as limits, in a suitable sense, of "regular" kernels. Our theory is based on Lebesgue-Stieltjes (Radon) integration, which appears to be an appropriate (in fact, necessary) tool for such problems. The developments are not of Fredholm type and they involve, in a very essential way, the use of an appropriate spectral theory. (Received April 3, 1941.)

Applied Mathematics

317. K. O. Friedrichs: *On the mathematical theory of the buckling of spherical shells.*

This paper is a contribution to von Kármán's recent nonlinear theory of buckling of spherical shells. The physically important state of equilibrium corresponds to a stationary value of the energy functional which is not a minimum but is of a degenerate character (no type number). The fact that buckling occurs only in the vicinity of one point is explained as a boundary layer phenomenon. For the analysis of the problem the Ritz method is employed. (Received April 2, 1941.)

318. F. G. Gravalos: *The algebraic integrals of Hill's equations.*

Hill's equations are $x'' - 2y' = 3x - \mu x/r^3$, $y'' + 2x' = -\mu y/r^3$, $r = (x^2+y^2)^{1/2}$, μ a parameter. Such a system of differential equations admits of Jacobi's integral $\frac{1}{2}(x'^2+y'^2) - \frac{3}{2}x^2 - \mu/r = c$. In this paper the theorem is proved: All the algebraic integrals of Hill's equations are functions of Jacobi's integral. The proof is done by parts in the form of three theorems. First, it is proved that all the integrals in the field $\Gamma(x', y', x, y, r)$ are reducible from Jacobi's. Second, the existence of integrals containing t rationally is ruled out. Finally, the above theorem is proved. Some parts of the proof are the same as those in a paper by C. L. Siegel on the restricted problem of three bodies (Transactions of this Society, vol. 39 (1936), pp. 225–237). (Received April 8, 1941.)

GEOMETRY

319. J. J. DeCicco: *Equilong geometry of series of collineal third order differential elements.*

The direct equilong group induces at a fixed line (u, v) of the plane a six-parameter group G_6 between the differential elements of third order. Any three elements (C_1, C_2, C_3) possess the fundamental invariant $(\delta_1 r_1+\delta_2 r_2+\delta_3 r_3)^2/(\delta_1 s_1+\delta_2 s_2+\delta_2 s_3)$, where δ_k is the distance from the point of C_i to that of C_j, r_i is the radius of curvature of C_i and $s_i=dr_i/d\theta_i$ is the rate of variation of the radius of curvature per unit radian measure of the inclination θ_i at the fixed line. In general, n elements possess $3n-6$ independent invariants, of which $n-1$ are distances, and the remaining $2n-5$ are of the new type given above. Any other invariant of these n elements must be a function of these $3n-6$ independent invariants. A series $r=r(w)$, $s=s(w)$ possesses the two fundamental differential invariants $k_1=(d^2r/dw^2)/(d^2r/dw^2)$, and $k_2=(d^2r/dw^2)^2 \div (d^2s/dw^2)$. Any two series with their curvatures the same functions of the distance w are equivalent under our group G_6. (Received May 21, 1941.)

320. Edward Kasner and J. J. DeCicco: *Conformal geometry of series of third order differential elements.*

Kasner and Comenetz have shown that at a fixed point of the plane the direct conformal group induces a six-parameter group G_6 between the differential elements of third order. In this paper it is found that three elements (C_1, C_2, C_3) possess the fundamental invariant $(k_1 \sin a_1+k_2 \sin a_2+k_3 \sin a_3)^3/(l_1 \sin 2a_1+l_2 \sin 2a_2+l_3 \sin 2a_3)$, where a_λ is the angle from C_μ to C_ν, k_λ is the curvature of C_λ, and $l_\lambda=dk_\lambda/ds_\lambda$ is the rate of variation of the curvature per unit length of arc at the fixed point. In general, n elements possess $3n-6$ independent invariants, of which $n-1$ are angles and the remaining $2n-5$ are of the new type given above. Any other invariant of these n elements must be a function of these $3n-6$ independent invariants. A series $k=k(\theta)$, $l=l(\theta)$ possesses the two fundamental differential invariants $\rho_1=(d^2k/d\theta^2+k) \div (d^2s/d\theta^2+dk/d\theta)$ and $\rho_2=(d^2k/d\theta^2+k)^2/(d^2l/d\theta^2+4l)$. Any two series with the curvatures ρ_1 and ρ_2 the same functions of the inclination θ are equivalent under the Kasner-Comenetz group G_6. (Received May 21, 1941.)

321. Edward Kasner and J. J. DeCicco: *General differential geometry of second order differential elements.*

Kasner (American Journal of Mathematics, vol. 28, pp. 203–213) showed that at a fixed point of the plane the entire group of arbitrary point transformations induces an eight-parameter group G_8 between differential elements of second order. He gave a complete discussion of all the invariants of n elements with all the appropriate geometric interpretations. This new paper considers the differential geometry of a series $q=q(p)$ under this group G_8, where $p=dy/dx$ and $q=d^2y/dx^2$. The length of arc of a series is $s=\int[5q^{iv}q^{v\,i}-6(q^v)^2]^{1/2}/q^{iv}dp$, and the curvature is $K=[25(q^{iv})^2q^{v\,ii} -105q^{iv}q^vq^{v\,i}+84(q^v)^3]/[5q^{iv}q^{v\,i}-6(q^v)^2]^{3/2}$, where the superscripts denote differentiation with respect to p. Any two series with their curvatures K the same functions of the arc length s are equivalent under the group G_8. (Received May 21, 1941.)

322. D. T. McClay: *Clifford numbers.*

The system of numbers $a+b\Lambda$, with a and b real and $\Lambda^2=1$, first used by Clifford, is an algebra of Weierstrass. If these "Clifford numbers" are represented by points

(a, b) in a rectangular coordinate system, the analytic functions correspond to transformations analogous to conformal transformations, the "angle" between two directions of slopes m_1, m_2 being defined as $\tanh^{-1}(m_2-m_1)/(1-m_1m_2)$. Lorentz transformations are a particular case. By adjoining suitably two intersecting lines at infinity, the plane of Clifford numbers may be mapped on a ring-shaped second order surface in projective 3-space just as complex numbers are mapped stereographically on a sphere. The linear transformations of a Clifford variable give geometrical transformations similar to the linear transformations of a complex variable, the metric dx^2-dy^2 replacing the euclidean. The exponential function $e^{x+y\Lambda}=e^x(\cosh y+\Lambda \sinh y)$ is not periodic, whereas $\sin(x+y\Lambda)=\sin x \cos y+\Lambda \cos x \sin y$ and the other trigonometric functions are doubly periodic. Cauchy's integral theorem and other complex variable theorems hold unchanged, or with slight modifications. (Received April 2, 1941.)

323. Nelson Robinson: *A transformation between osculating curves of a rational normal curve in an odd dimensional space.*

The equations of a rational normal curve Γ_n in a linear space of n dimensions are reduced to a canonical form. Hyperquadrics Q, containing Γ_n, and osculating curves Γ_{n-j} of Γ_n at a point P are defined geometrically and their equations derived. By use of a hyperquadric Q, a transformation involving a generalized null system is set up between osculating curves. This correspondence is proved to exist if and only if the ambient space is of odd dimensions. (Received May 14, 1941.)

Statistics and Probability

324. Alfred Basch: *A contribution to the theory of multiple correlation.*

If r_{zy} and r_{zz} are the correlation coefficients between x and y, and x and z, then r_{yz} lies between the limits $r_{zy}r_{zz}\pm k_{zy}k_{zz}$, where $k_{zy}=(1-r_{zy}^2)^{1/2}$, $k_{zz}=(1-r_{zz})^{1/2}$ are the alienation coefficients. In a geometrical representation where r_{zy} and r_{zz} are the rectangular coordinates, the ellipses inscribed in the square with the sides $r_{zy}=\pm 1$, $r_{zz}=\pm 1$ are the loci of constant lower and constant upper limits of r_{yz}. Rays parallel to the three coordinate axes give the three contour ellipses of the standard ellipsoid. From these three contour ellipses can be obtained the three correlation coefficients. If two of the contour ellipses are given, then the third must satisfy restricting conditions in agreement with the limits given for the third correlation coefficient. The intersection ellipses of the standard ellipsoid with the coordinate planes are characteristic for the coefficients of the correlation between two variables, freed from the influence of the third. The "limit cases" correspond to the degeneration of the standard ellipsoid, $r_{yz,\,z}=r_{zy,\,z}=r_{zz,\,y}$. In these limit cases there exists a functional relation between the three variables. The "middle case," $r_{yz}=r_{zy}r_{zz}$ is equivalent to $r_{yz,\,z}=0$. The standard ellipsoid intersects in this case the yz-plane in an ellipse, whose chief axes are the coordinate axes. (Received April 3, 1941.)

Topology

325. H. A. Arnold: *Homology in set-intersections, with an application to r-regular convergence.*

A and B are closed subsets of compact space R. Using Vietoris cycles the following lemmas are proved: (1) If the complete r-cycle γ^r, carried by A is ~ 0 in $A+B$, then

$A \cdot B$ contains a complete r-cycle μ^r (which may be the null cycle) with $\gamma^r \sim \mu^r$ in A. (2) If the complete r-cycle γ^r in $A \cdot B$ is ~ 0 in A and ~ 0 in B, and the $(r-1)$st Betti number of $A+B$ is 0, then $\gamma^r \sim 0$ in $A \cdot B$. These and other similar lemmas are used to prove that if a sequence of continua has a γ^1-continuum as a limit, then there can be no isolated points of non 0-regular convergence. (For 0-regular convergence, see G. T. Whyburn, Fundamenta Mathematicae, vol. 25 (1935), pp. 408–426.) A point of non 0-regular convergence is defined by the present writer to be a point of the limit set, on every neighborhood of which the condition of 0-regular convergence is violated. This theorem is extended to the cases $r>1$. (Received April 3, 1941.)

326. H. A. Arnold: *On r-regular convergence of sets.*

Using the theory of r-regular convergence, created by G. T. Whyburn (see Fundamenta Mathematicae, vol. 25 (1935), pp. 408–426), the following theorems are proved for compact spaces: (1) If the sequence $\{M_n\}$ of closed γ^r-continua M_n converges s-regularly to the limit set M, for all $s \leqq r$, then M is a γ^r-continuum. (2) If the sequence $\{M_n\}$ of closed sets M_n converges r-regularly to the limit set M, then for every sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets, such that the sequence $\{X_i\} = \{A_{n_i} \cdot B_{n_i}\}$ converges to X and the sequences $\{B_{n_i}\}$, $\{A_{n_i}\}$ converge $(r-1)$-regularly to B and A respectively, then $\{X_i\}$ converges $(r-1)$-regularly to X. (3) In the above notation, if $\{M_n\}$ converges to M, and $\{A_n \cdot B_n\}$ converges to $\{A \cdot B\}$, both r-regularly, then $\{A_n\}$ converges to A, and $\{B_n\}$ converges to B, both r-regularly. (Received April 3, 1941.)

327. T. A. Botts: *On convex sets in linear normed spaces.*

This note contains a simple proof of the theorem that in a linear normed space two convex bodies without common inner points are separated by a plane. (Received May 8, 1941.)

328. F. A. Ficken: *Certain systems of subsets of quasi and partially ordered sets.*

Let S be a set which is quasi ordered, partially ordered, or a lattice, and let $U(S)$ denote the set of subsets of S. Assuming the existence of the necessary meets and joins, a subset T which is a lattice is said to be a strict sublattice if both joins and meets in T of appropriate subsets of T agree with joins and meets in S, to be a JS-sublattice (MS-sublattice) if joins but not meets (meets but not joins) agree, and otherwise to be a loose sublattice. Subsystems of $U(S)$ are investigated which consist of those subsets of S which are: lower-convex, lower-normal, convex, strict-sublattices, MS-sublattices, and so on. Among the many sublattices L of $U(S)$ thus found are many which are MU-sublattices. Many of the L are sublattices of others of the L. It has been shown by Garrett Birkhoff that each quasi ordering of S determines and is determined by a complete strict sublattice of $U(S)$. The present discussion yields the specialization of this result to partial orderings, lattices, and so on. (Received April 7, 1941.)

329. Witold Hurewicz: *On duality theorems.*

Let A be a locally compact space, B a closed subset of A, and $H^n(A)$, $H^n(B)$, $H^n(A-B)$ the n-dimensional cohomology groups of the sets A, B and $A-B$ (with integers as coefficients). Consider "natural homomorphisms" $H^n(A) \to H^n(B) \to$

$H^{n+1}(A-B)\to H^{n+1}(A)\to H^{n+1}(A-B)$. It can be shown that the kernel of each of these homomorphisms is the image of the preceding homomorphism. This statement contains Kolmogoroff's generalization of Alexander's duality theorem and has many applications. Using the preceding theorem one can prove: If A and B are compact spaces of dimensions n and m respectively, the necessary and sufficient condition that the topological product $A\times B$ be of dimension $n+m$ is the existence of an open set $U\subset A$ and an open set $V\subset B$ such that $H^n(U)$ and $H^m(V)$ contain elements α and β satisfying the following conditions: If the integer d is a factor of the order of α, then $\beta\neq 0$ modulo d (that is, there is no element γ of $H^m(V)$ satisfying $\beta=d\gamma$); if the integer e is a factor of the order of β, then $\alpha\neq 0$ modulo e. (Received May 3, 1941.)

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